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Multivariate Linear Regression Model with Elliptically Contoured Distributed Errors and Monotone Missing Dependent Variables

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Abstract

In this paper, the multivariate linear regression model is studied under the assumptions that the error term of this model is described by the elliptically contoured distribution and the observations on the response variables are of a monotone missing pattern. It is primarily concerned with estimation of the model parameters', as well as, with the development of the likelihood ratio test in order to examine the existence of linear constraints on the regression coefficients. In this context, the multivariate linear regression model with the constant term as a sole explanatory variable is also studied and leads to estimators of the location and scale of elliptically contoured distributions with monotone missing data. A numerical example is presented for the explanation of the results.

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1 Introduction

Multivariate linear regression analysis is a well known statistical technique which helps to predict values of responses, dependent variables, from a set of explanatory, independent, variables. It is a popular statistical tool used in almost every branch of science and engineering. The classic linear multivariate regression model is analyzed assuming the error matrix has a multivariate normal distribution with zero mean matrix and a positive definite dispersion matrix. The role of the multivariate normal distribution is seminal in probability theory and statistics. However, many statistical papers and empirical studies show that the normal distribution is not capable of exhibiting important properties encountered in finance and economics, among other research areas. A well known insufficiency of the normal distributions are their light tails which fail to formulate for instance, observations of rates of return on common stock, according to Fama (1965) and Blattberg and Gonedes (1974). In this respect, there has been intense research in the use of

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nonnormal distributions in financial area. The papers by Zellner (1976) and Sutradhar and Ali (1986), include an extensive overview of relevant literature where the error term of the multivariate regression model can have nonnormal distributions and in particular t-distribution in practice. To tackle insufficiencies of normal distributions researchers focus on the broader class of elliptic distributions the last three decades. They provide useful alternatives to the multivariate normal distribution and many of the nice properties of the multivariate normal model holds for elliptic distributions. This generalized family of multivariate distributions, includes as representatives, the multivariate normal, multivariate t-distribution, Pearson type II and VII, multivariate symmetric Kotz type distribution. For a comprehensive monograph on elliptically contoured distributions see for example Fang and Zhang (1990), Fang *et al.* (1990) and Gupta and Varga (1993). Elliptic distributions and in particular the multivariate t-distribution have been considered by several authors to formulate the errors in the multivariate regression model. We refer, among others, to Zellner (1976), Sutradhar and Ali (1986), Galea *et al.* (1997), Liu (2002), Diaz-Garcia *et al.* (2003) and references therein.

In these and other treatments the multivariate linear regression model with nonnormal errors is studied under the assumption that complete data are available for the response and the explanatory variables. The investigation of this model in the case of incomplete data is particularly appealing from a theoretical as well as a practical viewpoint and it has occupied the literature of the subject. In this direction, Little (1992) and Rao and Toutenburg (1999) review the literature of regression analysis with missing values in the independent variables, while Robins and Rotnitzky (1995) discuss the semiparametric efficiency in multivariate regression models with missing data and in particular with monotone missing data for the response variable. Liu (1996) considers Bayesian estimation of multivariate linear regression models using fully observed explanatory variables and possible missing values from response variables. Tang *et al.* (2003) consider the same model with missing data in the response variables, when the nonresponse mechanism depends on the underlined values of the responses and hence is nonignorable. In a recent paper Raats *et al.* (2002) consider the problem of multivariate linear regression analysis, in the context of normally distributed error terms, for the specific case where the observations of the dependent variables appear a monotone missing pattern. Monotone missing data is a particular type of missing data which is common in practice (cf. Hao and Krishnamoorthy (2001)) and on the other hand, a non-monotone data set can be made monotone or nearly so by reordering the variables according to their missingness rates (cf. Schafer (1997), p. 218). There is an increasing interest in the development of statistical methods for handling monotone missing data from normal or elliptical populations (cf., for instance, Kanda and Fujikoshi (1998), Krishnamoorthy and Pannala (1998), Hao and Krishnamoorthy (2001), Chung and Han (2000), Batsidis and Zografos (2005) and references therein).

In this paper we extend the classic multivariate linear regression model in two aspects: on the one hand adopting elliptically contoured distributed errors and on the other hand considering monotone missing data for the response variables. More specifically, we consider a $p-$dimensional vector of response variables on a $q-$dimensional vector of explanatory variables when the explanatory are completely observed while the responses have missing values of a monotone pattern. These data are assumed to be missing completely at random (MCAR), that is the missing data mechanism can be ignored for inference (cf. Rubin (1976)). Practical examples of such data patterns, according to Raats *et al.* (2002), are experimental designs where new dependent variables are added during the experiment, panel surveys with drop outs or new members.

In the frame described previously, in the next section some preliminary concepts will be presented related to the elliptic family of distributions, monotone missing data and multivariate linear regression model. The necessary notation is also stated. In Section 3, the explicit form of the maximum likelihood estimators (MLE) will be derived for the parameters of the model. In Section 4, we will obtain the likelihood ratio test statistic in order to examine the existence of linear constraints on the regression parameters. In the final Section 5, we illustrate the results of this paper to a numerical example. In the Appendix, we deal with the special case of the constant term as sole explanatory variable. This case has been treated previously in the literature in the context of multivariate normal distribution by many authors (cf., for instance, Anderson (1957), Jinadasa and Tracy (1992), Fujisawa (1995)) and in the framework of elliptically contoured distribution by Batsidis and Zografos (2005). It will be shown that in this particular case of a constant term as sole explanatory variable, the main results of this paper lead to the similar ones mentioned above.

2 The model and Preliminaries

Let us suppose that the $N \times p$ random matrix $Y = (y_1, y_2, ..., y_N)^t$ has an elliptical distribution with an $N \times p$ location matrix μ and an $Np \times Np$ scale matrix $\Sigma \otimes I_N$, with $y_i \in R^p$, $i = 1, ..., N$, Σ a positive definite matrix of order p, I_N the identity matrix of order N and $\otimes$ denotes the Kronecker product of the respective matrices. Then, its density function is given by

$$|\Sigma|^{-N/2} f\left[tr\left\{\Sigma^{-1}(Y-\mu)^t(Y-\mu)\right\}\right], \tag{1}$$

where f is a one-dimensional real valued function such that (cf. Gupta and Varga (1993, p. 31))

$$0 \leq \int_0^\infty u^{(Np/2)-1} f(u)\, du < \infty.$$

We use in this case the notation $Y \sim EC_{N\times p}(\mu, \Sigma \otimes I_N)$, and we call $f(\cdot)$ the probability density function generator (p.d.f. generator).

Hence, we note (cf. Anderson *et al.* (1986)) that this distribution is written as a univariate elliptic distribution and then all the properties of univariate elliptic models, are still valid. Moreover, $y_1, y_2, ..., y_N$ can be considered as N uncorrelated realizations from a p-dimensional elliptic population (cf. Diaz-Garcia *et al.* (2003), Gupta and Varga (1993)) and relation (1) is in fact the likelihood function of them. We will present in the sequel some properties related to the univariate elliptic models.

Consider a $N \times p$ random matrix Y from an elliptically contoured distribution, as mentioned above, with unknown location matrix μ and unknown scale matrix $\Sigma \otimes I_N$, with Σ a positive definite matrix of order p. Let Y be partitioned as $(Y_1, Y_2, ..., Y_k)$, where Y_i is a $N \times p_i$ random matrix, $i = 1, ..., k$, and $p_1 + p_2 + ... + p_k = p$.

Following the notation of Kanda and Fujikoshi (1998), let the partitions of the location parameter μ and the scale matrix Σ, according to the ones of Y, be

$$\mu_{[i]} = (\mu_1, \mu_2, ..., \mu_i) \quad \text{and} \quad \Sigma_{(1,...,i)(1,...,i)} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdot & \cdot & \Sigma_{1i} \\ \Sigma_{21} & \Sigma_{22} & \cdot & \cdot & \Sigma_{2i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Sigma_{i1} & \Sigma_{i2} & \cdot & \cdot & \Sigma_{ii} \end{pmatrix}, \quad i = 1, ..., k,$$

where μ_j, is $N \times p_j$-dimensional and $\Sigma_{j\ell}$, are $p_j \times p_\ell$ matrices, with Σ_{jj} positive definite for $j, \ell = 1, ..., i$. Then, it is well known (cf. Fang and Zhang (1990), Gupta and Varga (1993)), that each Y_i, is distributed as an elliptically contoured $EC_{N\times p_i}(\mu_i, \Sigma_{ii} \otimes I_N)$, $i = 1, ..., k$.

Let now the transformation of the initial parameters μ and Σ to η and $\Delta = (\Delta_{ij})$, defined respectively by

$$\eta_1 = \mu_1, \ \eta_i = \mu_i - \mu_{[i-1]}\Delta_{(1...i-1)i}, \quad i = 2, ..., k,$$

$$\Delta_{11} = \Sigma_{11}, \ \Delta_{12} = \Delta_{21}^t = \Sigma_{11}^{-1}\Sigma_{12}, \ \Delta_{jj} = \Sigma_{jj} - \Sigma_{j(1...j-1)}\Sigma_{(1...j-1)(1...j-1)}^{-1}\Sigma_{(1...j-1)j},$$

$$\Delta_{(1...j-1)j} = \begin{pmatrix} \Delta_{1j} \\ \cdot \\ \cdot \\ \Delta_{j-1j} \end{pmatrix} = \Delta_{j(1...j-1)}^t = \Sigma_{(1...j-1)(1...j-1)}^{-1}\Sigma_{(1...j-1)j}, \tag{2}$$

for $j = 2, ..., k$. Under this notation we can easily seen that the conditional distribution of $Y_i|Y_{[i-1]}$, $Y_{[i-1]} = (Y_1, ..., Y_{i-1})$, is an elliptically contoured with a p.d.f. generator g_i, location parameter $\eta_i + y_{[i-1]}\Delta_{(1...i-1)i}$, $i = 2, ..., k$, and conditional covariance matrix $Cov\left(Y_i|Y_{[i-1]}\right)$, $i = 2, ..., k$, respectively

$$Cov\,(Y_2|Y_1) = \Delta_{22}h_2\left[tr\left\{\Delta_{11}^{-1}(Y_1 - \eta_1)^t(Y_1 - \eta_1)\right\}\right] \otimes I_N, \quad \text{and for } i = 3,...,k,$$

$$\begin{aligned} Cov\left(Y_i|Y_{[i-1]}\right) = (\Delta_{ii} \otimes I_N) \times h_i \Big\{ tr \big[\left(Y_1 - \eta_1 ... Y_{i-1} - \eta_{i-1} - Y_{[i-2]}\Delta_{(1...i-2)i-1}\right) \times \\ diag\left(\Delta_{11}^{-1}, ..., \Delta_{i-1i-1}^{-1}\right) \left(Y_1 - \eta_1 \ \ ... \ \ Y_{i-1} - \eta_{i-1} - Y_{[i-2]}\Delta_{(1...i-2)i-1} \right)^t \Big\}, \end{aligned} \tag{3}$$

where h_i, $i = 2, ..., k$, is a scalar function. This is a well known result appeared, for instance, in Fang *et al.* (1990, p. 45, 67) and Gupta and Varga (1993, p. 63). The expression for h_2 of specific elliptic models, like Pearson type VII etc., has been derived in Batsidis and Zografos (2005), for the case of univariate elliptic models.

Consider now the multivariate linear regression model with p dependent variables, q explanatory variables and N_1 items. This means that we consider the model

$$Y = XB + E, \tag{4}$$

where Y is a $N_1 \times p$ observation matrix of responses, X is a known $N_1 \times q$ model matrix of full column rank q, B is a $q \times p$ matrix of regression parameters with unknown values and E is a $N_1 \times p$ random matrix with $E = (\epsilon_1, \epsilon_2, ..., \epsilon_{N_1})^t$, where $\epsilon_i \in R^p$, $i = 1, ..., N_1$. The matrix E is known as error matrix. Further, we assume that the error matrix E has an elliptical distribution $EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$ and hence the density of Y is given by (1) with $\mu = XB$. This is typically called the elliptical multivariate linear regression model (cf. among others Diaz-Garcia *et al.* (2003)) and extends the respective classic linear model by using elliptic distribution for the error matrix E instead of multivariate normal distribution.

Motivated by the work of Raats *et al.* (2002), let us modify the classical elliptical multivariate linear regression model mentioned above. We assume that the observations of the dependent variables are incomplete and can be divided into k, $k \geq 2$, ordered groups according to the pattern of increasing missing rate. Group r contains p_r variables for which exactly the first N_r observations are available, $r = 1, ..., k$, with $N_1 > N_2 > ... > N_k$ and $p_1 + p_2 + ... + p_k = p$. Thus Y has the following form

$$Y = \begin{pmatrix} y_{11}^t & y_{21}^t & \cdot & \cdot & y_{k1}^t \\ y_{12}^t & y_{22}^t & \cdot & \cdot & y_{k2}^t \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_{1N_k}^t & y_{2N_k}^t & & & y_{kN_k}^t \\ \cdot & \cdot & & & \\ y_{1N_2}^t & y_{2N_2}^t & & & \\ \cdot & & & & \\ y_{1N_1}^t & & & & \end{pmatrix}, \tag{5}$$
$$\begin{matrix} \uparrow & \uparrow & & & \uparrow \\ Y_1 & Y_2 & & & Y_k \end{matrix}$$

where each y_{rj} is p_r-dimensional vector, $j = 1, ..., N_r$, $r = 1, ..., k$, with $p_1+p_2+...+p_k = p$ and we denote by $Y_r = (y_{r1}, y_{r2}, ..., y_{rN_r})^t$ the $N_r \times$ p_r matrix which contains all the available observations of group r, with $r = 1, ..., k$. Such a pattern is called k-step monotone missing pattern (cf. for example, Kanda and Fujikoshi (1998)).

We will present in the sequel the notation of this paper. The error matrix E is given by

$$E = \begin{pmatrix} \epsilon_{11}^t & \epsilon_{21}^t & \cdot & \epsilon_{k1}^t \\ \epsilon_{12}^t & \epsilon_{22}^t & \cdot & \epsilon_{k2}^t \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon_{1N_1}^t & \epsilon_{2N_1}^t & & \epsilon_{kN_1}^t \end{pmatrix}, \tag{6}$$

where each ϵ_{rj} is p_r-dimensional vector, $r = 1, ..., k$, $j = 1, ..., N_1$, with $p_1+p_2+...+p_k = p$. Similar to Y_r, we denote by E_r the $N_r \times$ p_r matrix given by $E_r = (\epsilon_{r1}, \epsilon_{r2}, ..., \epsilon_{rN_r})^t$,

$r = 1, ..., k$. We will denote by $Y_{(r-1)}$ and $E_{(r-1)}$ the $N_r \times p_{(r-1)}$ matrices, where $p_{(r-1)} = p_1 + ... + p_{r-1}$. These matrices contain the first N_r observations of the foregoings groups for $r = 2, ..., k$, while $Y_{(0)} = E_{(0)} = 0$. This means that

$$Y_{(r-1)} = \begin{pmatrix} y_{11}^t & \cdots & y_{r-1,1}^t \\ y_{12}^t & \cdots & y_{r-1,2}^t \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ y_{1N_r}^t & \cdots & y_{r-1,N_r}^t \end{pmatrix}, E_{(r-1)} = \begin{pmatrix} \epsilon_{11}^t & \cdots & \epsilon_{r-1,1}^t \\ \epsilon_{12}^t & \cdots & \epsilon_{r-1,2}^t \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \epsilon_{1N_r}^t & \cdots & \epsilon_{r-1,N_r}^t \end{pmatrix}, \ r = 2, ..., k. \quad (7)$$

Further let X_{lj} denotes the observed value of the l explanatory variable, $l = 1, ..., q$, for the j item, $j = 1, ..., N_1$. We assume that complete data are available for the explanatory variables and X is of the following form

$$X = \begin{pmatrix} X_{11} & X_{21} & \cdot & \cdot & X_{q1} \\ X_{12} & X_{22} & \cdot & \cdot & X_{q2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{1N_1} & X_{2N_1} & \cdot & \cdot & X_{qN_1} \end{pmatrix}. \quad (8)$$

Following the notation of Raats *et al.* (2002) let us denote by X_r the $N_r \times q$ matrix which contains the first N_r observations of all explanatory variables, $r = 1, ..., k$. According to this we have that

$$X_r = \begin{pmatrix} X_{11} & X_{21} & \cdot & \cdot & X_{q1} \\ X_{12} & X_{22} & \cdot & \cdot & X_{q2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{1N_r} & X_{2N_r} & \cdot & \cdot & X_{qN_r} \end{pmatrix}, \ r = 1, ..., k. \quad (9)$$

Moreover, we will assume that the matrix B of the regression parameters has the form

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdot & \cdot & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdot & \cdot & \beta_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{q1} & \beta_{q2} & \cdot & \cdot & \beta_{qk} \end{pmatrix} = (B_1, B_2, ..., B_k), \quad (10)$$

where each β_{lr}, is $1 \times p_r$, $l = 1, ..., q$, $r = 1, ..., k$, with $p_1 + p_2 + ... + p_k = p$ and B_r denotes the $q \times p_r$ submatrices of B. Denote also by $B_{(r-1)}$, the $q \times p_{(r-1)}$, $p_{(r-1)} = p_1 + ... + p_{r-1}$, submatrices of B, defined as follows

$$B_{(r-1)} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1,r-1} \\ \beta_{21} & \cdots & \beta_{2,r-1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \beta_{q1} & \cdots & \beta_{q,r-1} \end{pmatrix}, \ r = 2, ..., k. \quad (11)$$

Using the notation introduced by relations (5)-(11) the multivariate linear regression model (4) can be expressed in the following equivalent form

$$y_{rj}^t = \sum_{l=1}^{q} X_{lj}\beta_{lr} + \epsilon_{rj}^t, r = 1, ..., k, j = 1, ..., N_r,$$

$$Y_r = X_r B_r + E_r = \mu_r + E_r, \quad r = 1, ..., k, \tag{12}$$

and

$$Y_{(r-1)} = X_r B_{(r-1)} + E_{(r-1)} = \mu_{(r-1)} + E_{(r-1)}, \quad r = 2, ..., k. \tag{13}$$

3 Estimation of the model

In this section we will present the maximum likelihood approach for the estimation of the regression coefficients as well as the parameters of the elliptically contoured distribution in the presence of monotone missing data in the response variables. In particular, in the next subsection the maximum likelihood estimators (MLE) of B and Σ are derived. Consistent estimators of the parameters B, as well as, of the covariance matrix of the elliptically distributed error matrix E will be derived in Subsection 3.2 below.

3.1 Maximum likelihood estimators

Following the maximum likelihood approach we seek values of the unknown B and Σ that maximize the likelihood function. The likelihood function is the joint density of Y and it is denoted by $L_Y(B, \Sigma)$.

Theorem 1 *Consider the multivariate linear regression model given by (4), under the assumption that the error matrix E is distributed according to an elliptic distribution $EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$. On the basis of monotone missing pattern observations for the response variables of the form (5), the MLE of B and Σ are respectively*

$$\widehat{B} = \left(\widehat{B}_1, \widehat{B}_2, ..., \widehat{B}_k\right) \quad \textit{and} \quad \widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} & \cdot & \cdot & \widehat{\Sigma}_{1k} \\ \widehat{\Sigma}_{21} & \Sigma_{22} & \cdot & \cdot & \widehat{\Sigma}_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \widehat{\Sigma}_{k1} & \widehat{\Sigma}_{k2} & \cdot & \cdot & \widehat{\Sigma}_{kk} \end{pmatrix},$$

where

$$\widehat{B}_1 = (X_1^t X_1)^- X_1^t Y_1,$$

$$\begin{pmatrix} \widehat{B}_r \\ \widehat{\Delta}_{(1...r-1)r} \end{pmatrix} = \begin{pmatrix} X_r^t X_r & X_r^t e_{(r-1)} \\ e_{(r-1)}^t X_r & e_{(r-1)}^t e_{(r-1)} \end{pmatrix}^- \begin{pmatrix} X_r^t \\ e_{(r-1)}^t \end{pmatrix} Y_r, \quad \textit{for } r = 2, ..., k, \ k \geq 2,$$

with $e_{(r-1)} = Y_{(r-1)} - \widehat{\mu}_{(r-1)}$, *while*

$$\begin{aligned}
&\widehat{\Sigma}_{11} = \widehat{\Delta}_{11} = \lambda_{\max}(g_1)Q(\widehat{B}_1),\\
&\widehat{\Sigma}_{r(1\dots r-1)} = \widehat{\Delta}_{r(1\dots r-1)}\widehat{\Sigma}_{(1\dots r-1)(1\dots r-1)},\\
&\widehat{\Delta}_{rr} = \frac{\xi_{k,\max}(g_r)}{h_r}Q(\widehat{B}_r, \widehat{\Delta}_{(1\dots r-1)r})\\
&\widehat{\Sigma}_{rr} = \widehat{\Delta}_{rr} + \widehat{\Delta}_{r(1\dots r-1)}\widehat{\Sigma}_{(1\dots r-1)(1\dots r-1)}\widehat{\Delta}_{(1\dots r-1)r},
\end{aligned}$$

for $r = 2, ..., k$, $k \geq 2$, *where* $Q(\widehat{B}_1) = (Y_1 - X_1\widehat{B}_1)^t(Y_1 - X_1\widehat{B}_1)$, $Q(\widehat{B}_r, \widehat{\Delta}_{(1\dots r-1)r})$ *denotes the quantity*

$$\left(Y_r - \begin{pmatrix} X_r & e_{(r-1)} \end{pmatrix}\begin{pmatrix} \widehat{B}_r \\ \widehat{\Delta}_{(1\dots r-1)r} \end{pmatrix}\right)^t \left(Y_r - \begin{pmatrix} X_r & e_{(r-1)} \end{pmatrix}\begin{pmatrix} \widehat{B}_r \\ \widehat{\Delta}_{(1\dots r-1)r} \end{pmatrix}\right)$$

and h_r, *the function related to the conditional covariance matrix* $Cov\left(Y_r|Y_{(r-1)}\right)$, $r = 2, ..., k$, $k \geq 2$, *given by (3). Moreover,* g_1 *and* g_r, $r = 2, ..., k$, *are the nonincreasing, by assumption, p.d.f. generators respectively, of the marginal density of* Y_1 *and the conditional density* $Y_r|Y_{(r-1)}$, *while* $\lambda_{\max}(g_1)$ *denotes the point at which the function* $\lambda^{-N_1p_1/2}g_1(p_1/\lambda)$ *arrives at its maximum and* $\xi_{r,\max}(g_r)$ *the point at which the function* $\xi^{-N_rp_r/2}g_r(p_r/\xi)$ *arrives at its maximum,* $r = 2, ..., k$, $k \geq 2$.

Proof. The proof follows the conditional likelihood approach introduced by Anderson (1957). Writing the joint density as the product of the marginal and conditional densities functions and taking into account relation (1) and the reparametrization (2), we can express the likelihood function as follows

$$L_Y(\mu, \Sigma) = L_{Y_1}(\eta_1, \Delta_{11})L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22})...L_{Y_k|Y_{(k-1)}}(\eta_k, \Delta_{(1\dots k-1)k}, \Delta_{kk}). \tag{14}$$

In view of (2), there is a one-to-one correspondence (cf. Anderson (1957), Little and Rubin (2002, p. 135)) between the initial (μ, Σ) and the natural parameters (η, Δ) in the conditional approach. Therefore, it is enough to derive the MLE of (η, Δ). We will obtain, at the beginning, the MLE of η_1 and Δ_{11} based on L_{Y_1}, and then, replacing in the expression of $L_{Y_2|Y_{(1)}}$, η_1 and Δ_{11} by their MLE, we will derive the MLE of η_2, Δ_{12} and Δ_{22} based on the conditional likelihood. We repeat this procedure until the last part of the product given by the right-side of relation (14). Therefore, the MLE of η_1 and Δ_{11} will be obtained, at the beginning, by the maximization, with respect to η_1 and Δ_{11}, of the first part of the likelihood which, using relation (1), is given by

$$L_{Y_1}(\eta_1, \Delta_{11}) = |\Delta_{11}|^{-N_1/2}g_1\left(tr\left\{\Delta_{11}^{-1}(Y_1 - \eta_1)^t(Y_1 - \eta_1)\right\}\right), \tag{15}$$

where g_1, is the nonincreasing, by assumption, p.d.f. generator function of the marginal density of Y_1. By monotonicity of g_1, for a given $\Delta_{11} > 0$, we have to minimize according to η_1, the quantity $(Y_1 - \eta_1)^t(Y_1 - \eta_1)$. Thus, based on the relations (2) and (12), we have to minimize with respect to B_1, the quantity $(Y_1 - X_1B_1)^t(Y_1 - X_1B_1)$. This quantity is minimized by $\widehat{B}_1 = (X_1^tX_1)^-X_1^tY_1$.

Hence the concentrated likelihood is

$$L_{Y_1}(\widehat{B}_1, \Delta_{11}) = |\Delta_{11}|^{-N_1/2} g_1 \left[tr(\Delta_{11}^{-1} Q(\widehat{B}_1)) \right],$$

with

$$Q(\widehat{B}_1) = (Y_1 - X_1\widehat{B}_1)^t (Y_1 - X_1\widehat{B}_1). \tag{16}$$

Following now the steps of the proof of Theorem 4.1.1, of Fang and Zhang (1990), the MLE of Δ_{11} is given by

$$\widehat{\Delta}_{11} = \lambda_{\max}(g_1) Q(\widehat{B}_1). \tag{17}$$

We will now concentrate on the MLE of the parameters η_2, Δ_{21} and Δ_{22}. The conditional likelihood $L_{Y_2|Y_{(1)}} = L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22})$, in view of relations (1) and (3), is defined by

$$\begin{aligned} L_{Y_2|Y_{(1)}} = \; & |\Delta_{22} h_2|^{-N_2/2} \\ & \times g_2 \left\{ tr \left[(h_2 \Delta_{22})^{-1} (Y_2 - \eta_2 - Y_{(1)}\Delta_{12})^t (Y_2 - \eta_2 - Y_{(1)}\Delta_{12}) \right] \right\} \end{aligned} \tag{18}$$

with h_2 related to the conditional covariance matrix $Cov\,(Y_2|Y_1)$, defined by (3). If we replace in the expression of $L_{Y_2|Y_{(1)}}$, η_1 and Δ_{11}, by their MLE, then using monotonicity of g_2, the maximum of $L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22})$ with respect to η_2 and Δ_{12} arrives at the values of η_2 and Δ_{12} which minimize the quantity

$$(Y_2 - \eta_2 - Y_{(1)}\Delta_{12})^t (Y_2 - \eta_2 - Y_{(1)}\Delta_{12}).$$

From relation (2), we have that $\eta_2 = \mu_2 - \mu_{(1)}\Delta_{12}$. Hence, if we replace $\mu_{(1)}$ by its MLE we have to minimize with respect to μ_2 and Δ_{12} the quantity

$$(Y_2 - \mu_2 - (Y_{(1)} - \widehat{\mu}_{(1)})\Delta_{12})^t (Y_2 - \mu_2 - (Y_{(1)} - \widehat{\mu}_{(1)})\Delta_{12}).$$

Using the fact that $\mu_2 = X_2 B_2$, we have to minimize with respect to B_2 and Δ_{12} the quantity

$$(Y_2 - X_2 B_2 - (Y_{(1)} - \widehat{\mu}_{(1)})\Delta_{12})^t (Y_2 - X_2 B_2 - (Y_{(1)} - \widehat{\mu}_{(1)})\Delta_{12}),$$

or equivalently the quantity

$$\left(Y_2 - \begin{pmatrix} X_2 & e_{(1)} \end{pmatrix} \begin{pmatrix} B_2 \\ \Delta_{12} \end{pmatrix} \right)^t \left(Y_2 - \begin{pmatrix} X_2 & e_{(1)} \end{pmatrix} \begin{pmatrix} B_2 \\ \Delta_{12} \end{pmatrix} \right),$$

where $e_{(1)} = Y_{(1)} - \widehat{\mu}_{(1)}$. After some algebra we obtain that

$$\begin{pmatrix} \widehat{B}_2 \\ \widehat{\Delta}_{12} \end{pmatrix} = \begin{pmatrix} X_2^t X_2 & X_2^t e_{(1)} \\ e_{(1)}^t X_2 & e_{(1)}^t e_{(1)} \end{pmatrix}^{-} \begin{pmatrix} X_2^t \\ e_{(1)}^t \end{pmatrix} Y_2. \tag{19}$$

In order to derive the MLE of the parameter Δ_{22}, we have to maximize the quantity

$$L_{Y_2|Y_{(1)}}(\widehat{\eta}_2, \widehat{\Delta}_{21}, \Delta_{22}) = \left| h_2(\widehat{\eta}_1, \widehat{\Delta}_{11}) \times \Delta_{22} \right|^{-N_2/2} g_2 \left\{ tr \left[\left(h_2(\widehat{\eta}_1, \widehat{\Delta}_{11}) \times \Delta_{22} \right)^{-1} Q(\widehat{B}_2, \widehat{\Delta}_{12}) \right] \right\},$$

where

$$Q(\widehat{B}_2,\widehat{\Delta}_{12})=\left(Y_2-\left(\begin{array}{cc} X_2 & e_{(1)}\end{array}\right)\left(\begin{array}{c}\widehat{B}_2\\ \widehat{\Delta}_{12}\end{array}\right)\right)^t\left(Y_2-\left(\begin{array}{cc} X_2 & e_{(1)}\end{array}\right)\left(\begin{array}{c}\widehat{B}_2\\ \widehat{\Delta}_{12}\end{array}\right)\right). \quad (20)$$

Following again the steps of the proof of Theorem 4.1.1, of Fang and Zhang (1990), we obtain the MLE of $\Delta_{22}=\Sigma_{2\cdot 1}$, that is

$$\widehat{\Delta}_{22}=\widehat{\Sigma}_{2\cdot 1}=\frac{\xi_{2,\max}(g_2)}{h_2(\widehat{\eta}_1,\widehat{\Delta}_{11})}Q(\widehat{B}_2,\widehat{\Delta}_{12}). \quad (21)$$

If we repeat the same procedure, the last term $L_{Y_k|Y_{(k-1)}}=L_{Y_k|Y_{(k-1)}}(\eta_k,\Delta_{(1\ldots k-1)k},\Delta_{kk})$, of the likelihood function (14) becomes

$$\begin{array}{l} L_{Y_k|Y_{(k-1)}}=|\Delta_{kk}h_k|^{-N_k/2}\times \\ g_k\left[tr\left\{(\Delta_{kk}h_k)^{-1}(Y_k-\eta_k-Y_{(k-1)}\Delta_{(1\ldots k-1)k})^t(Y_k-\eta_k-Y_{(k-1)}\Delta_{(1\ldots k-1)k})\right\}\right], \end{array} \quad (22)$$

where the scalar functions g_k and h_k are respectively the p.d.f. generator of $Y_k|Y_{(k-1)}$, and the function related to the conditional covariance matrix of $Y_k|Y_{(k-1)}$, $k\geq 2$, defined in (3).
In a similar manner, as in the case of the maximization of $L_{Y_2|Y_{(1)}}$, we obtain the MLE estimator of $\left(\begin{array}{c} B_k\\ \Delta_{(1\ldots k-1)k}\end{array}\right)$ to be

$$\left(\begin{array}{cc} X_k^tX_k & X_k^te_{(k-1)}\\ e_{(k-1)}^tX_k & e_{(k-1)}^te_{(k-1)}\end{array}\right)^-\left(\begin{array}{c} X_k^t\\ e_{(k-1)}^t\end{array}\right)Y_k, \quad (23)$$

where $e_{(k-1)}=Y_{(k-1)}-\widehat{\mu}_{(k-1)}$. Moreover, we have that

$$\widehat{\Delta}_{kk}=\frac{\xi_{k,\max}}{h_k}Q(\widehat{B}_k,\widehat{\Delta}_{(1\ldots k-1)k}), \quad (24)$$

for $k\geq 3$, with $\xi_{k,\max}$ being the point at which $\xi^{-N_kp_k/2}g_k(p_k/\xi)$ arrives at its maximum and $Q(\widehat{B}_k,\widehat{\Delta}_{(1\ldots k-1)k})$ is

$$\left(Y_k-\left(\begin{array}{cc} X_k & e_{(k-1)}\end{array}\right)\left(\begin{array}{c}\widehat{B}_k\\ \widehat{\Delta}_{(1\ldots k-1)k}\end{array}\right)\right)^t\left(Y_k-\left(\begin{array}{cc} X_k & e_{(k-1)}\end{array}\right)\left(\begin{array}{c}\widehat{B}_k\\ \widehat{\Delta}_{(1\ldots k-1)k}\end{array}\right)\right). \quad (25)$$

Based on the previous discussion, the MLE $\widehat{\mu}$ and $\widehat{\Sigma}$ of the initial parameters μ and Σ, can be obtained by using relation (2). Hence, using the relations

$$\begin{array}{l} \widehat{\Sigma}_{11}=\widehat{\Delta}_{11},\\ \widehat{\Sigma}_{21}=\widehat{\Delta}_{21}\widehat{\Sigma}_{11},\\ \widehat{\Sigma}_{22}=\widehat{\Sigma}_{2\cdot 1}+\widehat{\Sigma}_{21}\widehat{\Sigma}_{11}^{-1}\widehat{\Sigma}_{12}=\widehat{\Delta}_{22}+\widehat{\Delta}_{21}\widehat{\Delta}_{11}\widehat{\Delta}_{12},\\ \widehat{\Sigma}_{k(1\ldots k-1)}=\widehat{\Delta}_{k(1\ldots k-1)}\widehat{\Sigma}_{(1\ldots k-1)(1\ldots k-1)},\quad k\geq 3,\\ \widehat{\Sigma}_{kk}=\widehat{\Delta}_{kk}+\widehat{\Delta}_{k(1\ldots k-1)}\widehat{\Sigma}_{(1\ldots k-1)(1\ldots k-1)}\widehat{\Delta}_{(1\ldots k-1)k},\quad k\geq 3, \end{array}$$

we obtain the desired results. ■

Remark 1. a) An application of Theorem 1 for $N_1 = N_2 = ... = N_k$, leads to the MLE of B and Σ of the elliptic multivariate linear regression model, in the complete data case (cf. Diaz-Garcia *et al.* (2003)).
b) Taking into account Theorem 1, we can obtain after some algebra, the following equivalent expressions for the MLE of $\Delta_{(1...r-1)r}$ and B_r,

$$\widehat{\Delta}_{(1...r-1)r} = \left(e^t_{(r-1)} U_r e_{(r-1)}\right)^- e^t_{(r-1)} U_r Y_r \tag{26}$$

and

$$\widehat{B}_r = \left(X_r^t X_r\right)^- X_r^t \left[Y_r - e_{(r-1)} \widehat{\Delta}_{(1...r-1)r}\right] \tag{27}$$

for $r = 2, ..., k$, $k \geq 2$, with $U_r = I - X_r\left(X_r^t X_r\right)^- X_r^t$. The above expressions will be used later in the study of the consistency property of the MLE of Theorem 1.

A particular multivariate linear regression model is the constant term model obtained from (4) when $X = 1_{N_1}$, with 1_{N_1} the $N_1 \times 1$ unity vector and $B = (B_1, B_2, ..., B_k)$, where each B_i is $1 \times p_i$ dimensional with $p_1 + p_2 + ... + p_k = p$. The $N_1 \times p$ error random matrix E is supposed again elliptically distributed $EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$. In other words, we assume the elliptical multivariate linear regression model with the constant term as explanatory variable, under the extra assumption that there exist monotone missing data for the response variables.

This model has received a lot of attention in literature under the assumption of multivariate distributed error terms. We refer among others to Anderson (1957), Jinadasa and Tracy (1992), Fujisawa (1995) and references therein. These results have been also obtained by Raats *et al.* (2002). Our aim is to prove that the MLE for the regression coefficients, as well as, for Σ obtained previously reduce to the same expressions determined recently by Batsidis and Zografos (2005) in the case of elliptically contoured distributions.

In the sequel we will denote

$$\begin{aligned}
&\overline{Y}_r = \tfrac{1}{N_r}\sum_{\nu=1}^{N_k} y^t_{r\nu} = \tfrac{1}{N_r} 1^t_{N_r} Y_r \text{ and } S_{rr,r} = (Y_r - 1_{N_r}\overline{Y}_r)^t (Y_r - 1_{N_r}\overline{Y}_r),\ r \geq 1 \\
&\overline{Y}_{(r-1)} = \tfrac{1}{N_r} 1^t_{N_r} Y_{(r-1)} \text{ and } S_{(1...r-1)r,r} = \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right)^t \left(Y_r - 1_{N_r}\overline{Y}_r\right),\ r \geq 2 \\
&S_{(1...r-1)(1...r-1),r} = \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right)^t \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right),\ r \geq 2 \\
&S_{r\cdot(1...r-1),r} = S_{rr,r} - S_{r(1...r-1),r} S^{-1}_{(1...r-1)(1...r-1),r} S_{(1...r-1)r,r},\ r \geq 2.
\end{aligned}$$

In the next proposition we obtain the estimators of the constant term model. The proof of the proposition is outlined in the Appendix.

Proposition 2 *Consider the multivariate linear regression model given by (4), with the constant term as a sole explanatory variable, under the assumption that the error matrix E is distributed according to an elliptical distribution $EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$. On the basis of a monotone missing pattern observations for the response variables, the MLE of B and Σ are* *respectively* $\widehat{B} = \left(\widehat{B}_1, \widehat{B}_2, ..., \widehat{B}_k\right)$ *and* $\widehat{\Sigma} = \begin{pmatrix} \widehat{\Sigma}_{11} & \widehat{\Sigma}_{12} & \cdot & \cdot & \widehat{\Sigma}_{1k} \\ \widehat{\Sigma}_{21} & \widehat{\Sigma}_{22} & \cdot & \cdot & \widehat{\Sigma}_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \widehat{\Sigma}_{k1} & \widehat{\Sigma}_{k2} & \cdot & \cdot & \widehat{\Sigma}_{kk} \end{pmatrix}$, *where* $\widehat{B}_1 = \overline{Y}_1$ *and* $\widehat{B}_r = \overline{Y}_r - \left(\overline{Y}_{(r-1)} - \widehat{B}_{(r-1)}\right)\widehat{\Delta}_{(1...r-1)r}$, *with* $\widehat{\Delta}_{(1...r-1)r} = S^{-1}_{(1...r-1)(1...r-1),r} S_{(1...r-1)r,r}$, *for* $r = 2, ..., k$, *while* $\widehat{\Sigma}_{11} = \widehat{\Delta}_{11} = \lambda_{\max}(g_1)S_{11,1}$ *and* $\widehat{\Sigma}_{rr}$ *is given by the relation* $\widehat{\Sigma}_{rr} = \widehat{\Delta}_{rr} + \widehat{\Delta}_{r(1...r-1)}\widehat{\Sigma}_{(1...r-1)(1...r-1)}\widehat{\Delta}_{(1...r-1)r}$, *for* $r = 2, ..., k$, *where* $\widehat{\Delta}_{rr} = \frac{\xi_{r,\max}}{h_r} S_{r\cdot(1...r-1),r}$, *with* $g_1, h_r, g_r, \lambda_{\max}, \xi_{r,\max}$ *as given in Theorem 1, for* $r = 2, ..., k$.

3.2 The consistency property

Consistency refers to a limiting property of an estimator and it is usually considered a basic requirement of an inference procedure. Our object in this subsection is to derive the consistent estimators of the parameters B of the model (4), as well as, of the covariance matrix of the elliptically distributed error matrix E of this model. Similar work has been made previously by Sutradhar and Ali (1986) for multivariate $t-$distributed errors in the model (4) and complete data for the responses. Recently, Raats *et al.* (2004) derived the respective consistent estimators under the assumption of normally distributed error matrix and monotone missing observations for the responses Y of the model (4). In this respect the results of this subsection generalize in both aspects the results of the above mentioned papers.

It will be shown, in the sequel, that the MLE $\widehat{B}$ derived in Theorem 1 is a consistent estimator of the parameters B of the model (4). A similar conclusion it is not true in general for the MLE $\widehat{\Sigma}$ of the scale matrix Σ of the elliptically distributed error matrix E. This point will be clarified later on in a remark at the end of this subsection. We will use the symbol $\underset{N_i \to \infty}{p\lim}$ to denote converge in probability, as $N_i \to \infty$, $i = 1, ..., k$. Convergence in probability of random matrices is considered in the element-wise sense.

Consider the multivariate linear regression model (4), that is, $Y = XB + E$ and suppose, as above, that $E \sim EC_{N_1 x p}(0, \Sigma \otimes I_{N_1})$. The covariance matrix of E is $Cov(E) = \Omega \otimes I_{N_1}$, where Ω is a positive definite $p \times p$ matrix related with the scale matrix by the equality $\Omega = c\Sigma$, where $c = -2\Psi'(0)$ is a constant which depends on the characteristic function Ψ of E (cf., for instance, Gupta and Varga (1993, p. 33)). Let the following partition for the covariance matrix Ω:

$$\Omega_{(1,...,i)(1,...,i)} = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdot & \cdot & \Omega_{1i} \\ \Omega_{21} & \Omega_{22} & \cdot & \cdot & \Omega_{2i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Omega_{i1} & \Omega_{i2} & \cdot & \cdot & \Omega_{ii} \end{pmatrix} = c\Sigma_{(1,...,i)(1,...,i)}, \quad i = 1, ..., k,$$

similar to the partition defined previously for the matrix Σ. Further, denote by

$$\Omega_{(1\ldots r-1)r} = \Omega^t_{r(1\ldots r-1)} = \begin{pmatrix} \Omega_{1r} \\ \Omega_{2r} \\ \cdot \\ \cdot \\ \Omega_{r-1,r} \end{pmatrix} = c\Sigma_{(1\ldots r-1)r}, \ r = 2, \ldots, k.$$

Under these circumstances the consistent estimators of B and Ω are derived in the next theorem. The keynote in the proof of the theorem is the convergence in probability of submatrices of the error matrix E to suitable submatrices of Ω. Although it is immediate in the case of a multivariate normal distributed error matrix E, in view of the weak law of large numbers, it is not so obvious in the case of an elliptic error matrix E considered here. The next lemma establishes the convergence in probability of submatrices of E. The results of the lemma are proved under the additional assumption that the elliptic distribution $EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$ of E possesses a consistency property as it is defined and studied by Kano (1994). Consistency property of the elliptical density function of E is equivalent, according to Theorem 1 of Kano (1994), to the fact that the characteristic function of E and hence the constant c, which appears in the $Cov(E) = (c\Sigma) \otimes I_{N_1}$, does not depend on the dimension of the distribution. This assumption permits to prove the following lemma.

Lemma 3 *Consider the multivariate linear regression model (4) and suppose that the density function of the error matrix $E \sim EC_{N_1 \times p}(0, \Sigma \otimes I_{N_1})$, obeys the consistency property of Kano (1994). Then, under the assumption that $\lim_{N_i \to \infty} \left(\frac{1}{N_i} X_i^t X_i\right)^-$ exists, $i = 1, \ldots, k$,*

i) $p\lim_{N_r \to \infty} \left(\frac{1}{N_r} E^t_{(r-1)} X_r\right) = 0,$

ii) $p\lim_{N_r \to \infty} \left(\frac{1}{N_r} E^t_{(r-1)} E_{(r-1)}\right) = c\Sigma_{(1\ldots r-1)(1\ldots r-1)},$

iii) $p\lim_{N_r \to \infty} \left(\frac{1}{N_r} E^t_{(r-1)} E_r\right) = c\Sigma_{(1\ldots r-1)r}.$

Proof. We will prove part ii). The proof of parts i) and iii) are similar and they are omitted. If $\left(E^t_{(r-1)} E_{(r-1)}\right)_{ij}$ denotes the (i, j) submatrix of $E^t_{(r-1)} E_{(r-1)}$, then

$$\frac{1}{N_r} \left(E^t_{(r-1)} E_{(r-1)}\right)_{ij} = \frac{1}{N_r} \sum_{l=1}^{N_r} \epsilon_{il} \epsilon^t_{jl}, \ r = 2, \ldots, k.$$

We observe that $\epsilon_{il} \epsilon^t_{jl}$, $l = 1, \ldots, N_r$, are uncorrelated $p_i \times p_j$ random matrices with $E\left(\epsilon_{il} \epsilon^t_{jl}\right) = Cov\left(\epsilon_{il} \epsilon^t_{jl}\right) = \Omega_{ij} = c\Sigma_{ij}$, where c does not depend on the sample size N_r, taking into account the consistency, in Kano's (1994) sense, of the density of the error matrix E. If we apply the weak law of large numbers (cf. Rao (1973, p. 112)) to the sequence of random matrices $\epsilon_{i1} \epsilon^t_{j1}, \epsilon_{i2} \epsilon^t_{j2}, \ldots, \epsilon_{iN_r} \epsilon^t_{jN_r}$, we have that

$$p\lim_{N_r \to \infty} \left\{ \frac{1}{N_r} \left(E^t_{(r-1)} E_{(r-1)}\right)_{ij} \right\} = p\lim_{N_r \to \infty} \left\{ \frac{1}{N_r} \sum_{l=1}^{N_r} \epsilon_{il} \epsilon^t_{jl} \right\} = \Omega_{ij} = c\Sigma_{ij},$$

hence

$$\underset{N_r\to\infty}{p\lim}\left\{\frac{1}{N_r}\left(E^t_{(r-1)}E_{(r-1)}\right)\right\} = \Omega_{(1\ldots r-1)(1\ldots r-1)} = c\Sigma_{(1\ldots r-1)(1\ldots r-1)},$$

which proves the desired result. ■

Lemma 4 *Under the assumptions of Lemma 3,*

$$\underset{N_r\to\infty}{p\lim}\widehat{\Delta}_{(1\ldots r-1)r} = \underset{N_r\to\infty}{p\lim}\left\{\left(\tfrac{1}{N_r}E^t_{(r-1)}E_{(r-1)}\right)^-\left(\tfrac{1}{N_r}E^t_{(r-1)}E_r\right)\right\}.$$

Proof. Using relation (26), we obtain after some algebra that

$$\widehat{\Delta}_{(1\ldots r-1)r} = \left(E^t_{(r-1)}U_rE_{(r-1)}\right)^- E^t_{(r-1)}U_rE_r, \text{ for } r = 2, \ldots, k,$$

where $U_r = I - X_r(X_r^tX_r)^-X_r^t = I - H_r$. Hence,

$$\underset{N_r\to\infty}{p\lim}\widehat{\Delta}_{(1\ldots r-1)r} = \underset{N_r\to\infty}{p\lim}\left\{\left(\frac{1}{N_r}E^t_{(r-1)}U_rE_{(r-1)}\right)^-\right\}\underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}U_rE_r\right). \tag{28}$$

Using the fact that $U_r = I - H_r$, we obtain that

$$\begin{aligned}\underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}U_rE_r\right) &= \underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}E_r\right) - \underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}H_rE_r\right)\\ &= \underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}E_r\right),\end{aligned} \tag{29}$$

because of

$$\begin{aligned}\underset{N_r\to\infty}{p\lim}\left(\tfrac{1}{N_r}E^t_{(r-1)}H_rE_r\right) &= \underset{N_r\to\infty}{p\lim}\left\{\tfrac{1}{N_r}E^t_{(r-1)}X_r\right\}\underset{N_r\to\infty}{\lim}\left\{\left(\tfrac{1}{N_r}X_r^tX_r\right)^-\right\}\\ &\times\underset{N_r\to\infty}{p\lim}\left\{\tfrac{1}{N_r}X_r^tE_r\right\} = 0,\end{aligned}$$

in view of Lemma 3 i). Moreover, taking into account again Lemma 3 i)

$$\underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}H_rE_{(r-1)}\right) = 0,$$

and hence

$$\underset{N_r\to\infty}{p\lim}\left(\frac{1}{N_r}E^t_{(r-1)}U_rE_{(r-1)}\right) = \underset{N_r\to\infty}{p\lim}\left\{\frac{1}{N_r}E^t_{(r-1)}E_{(r-1)}\right\}. \tag{30}$$

Therefore from (28), taking into consideration the intermediate results (29) and (30), we obtain that

$$\underset{N_r\to\infty}{p\lim}\widehat{\Delta}_{(1\ldots r-1)r} = \underset{N_r\to\infty}{p\lim}\left\{\left(\frac{1}{N_r}E^t_{(r-1)}E_{(r-1)}\right)^-\left(\frac{1}{N_r}E^t_{(r-1)}E_r\right)\right\},$$

which is the desired result. ■

We are now ready to derive the consistent estimators of B and Ω in the next theorem.

Theorem 5 *Under the assumption that* $\lim_{N_i\to\infty}\left\{\left(\frac{1}{N_i}X_i^tX_i\right)^-\right\}$ *exists we have that:*

a) $p\lim_{N_r\to\infty}\widehat{\Delta}_{(1...r-1)r} = \Delta_{(1...r-1)r}$, *for* $r = 2, ..., k$

b) $p\lim_{N_i\to\infty}\widehat{B}_i = B_i$, *for* $i = 1, ..., k$ *and*

c) $p\lim_{N_1\to\infty}\widetilde{\Delta}_{11} = \Omega_{11}$, *while* $p\lim_{N_r\to\infty}\widetilde{\Delta}_{rr} = \Omega_{rr} - \Omega_{r(1...r-1)}\Omega^{-1}_{(1...r-1)(1...r-1)}\Omega_{(1...r-1)r}$,

with $\widetilde{\Delta}_{11} = \frac{1}{N_1}Q(\widehat{B}_1)$ *and* $\widetilde{\Delta}_{rr} = \frac{1}{N_r}Q(\widehat{B}_r, \widehat{\Delta}_{(1...r-1)r})$, *for* $r = 2, ..., k$. *Moreover,* $\widehat{B} = \left(\widehat{B}_1, \widehat{B}_2, ..., \widehat{B}_k\right)$ *and* $\widehat{\Delta}_{(1...r-1)r}$, *as well as,* $Q(\widehat{B}_1)$ *and* $Q(\widehat{B}_r, \widehat{\Delta}_{(1...r-1)r})$, $r = 2, ..., k$, *are defined in Theorem 1.*

Proof. a) Based on Lemma 3 ii) and iii) it can be easily seen that

$$\begin{aligned} p\lim_{N_r\to\infty}\left\{\left(\frac{1}{N_r}E^t_{(r-1)}E_{(r-1)}\right)^-\left(\frac{1}{N_r}E^t_{(r-1)}E_r\right)\right\} &= \Sigma^{-1}_{(1...r-1)(1...r-1)}\Sigma_{(1...r-1)r} \\ &= \Delta_{(1...r-1)r}. \end{aligned} \tag{31}$$

Relation (31) and Lemma 4 complete the proof of part a).

b) Motivated by Raats *et al.* (2004), we will prove that $p\lim_{N_i\to\infty}\widehat{B}_i = B_i$, for $i = 1, ..., k$, by using an induction argument. For $i = 1$, we have that

$$\widehat{B}_1 = (X_1^tX_1)^-X_1^tY_1$$

and taking into consideration (12) we obtain

$$\widehat{B}_1 = (X_1^tX_1)^-X_1^t(X_1B_1 + E_1) = B_1 + (X_1^tX_1)^-X_1^tE_1.$$

Hence

$$\begin{aligned} p\lim_{N_1\to\infty}\left(\widehat{B}_1 - B_1\right) &= p\lim_{N_1\to\infty}\left\{\left(\frac{1}{N_1}X_1^tX_1\right)^-\frac{1}{N_1}X_1^tE_1\right\} \\ &= \lim_{N_1\to\infty}\left(\frac{1}{N_1}X_1^tX_1\right)^- p\lim_{N_1\to\infty}\left(\frac{1}{N_1}X_1^tE_1\right) = 0, \end{aligned}$$

in view of Lemma 3 i).

Afterwards, using the induction assumption that $p\lim_{N_i\to\infty}\widehat{B}_i = B_i$, for $i = 1, ..., k-1$, which implies that $p\lim_{N_k\to\infty}\widehat{B}_{(k-1)} = B_{(k-1)}$, we are going to prove that $p\lim_{N_k\to\infty}\widehat{B}_k = B_k$. Based on relation (27) and taking into account (12), we have that

$$\widehat{B}_k = B_k + \left(X_k^tX_k\right)^- X_k^t\left[E_k - e_{(k-1)}\widehat{\Delta}_{(1...k-1)k}\right],$$

or equivalently

$$\widehat{B}_k - B_k = \left(X_k^t X_k\right)^- X_k^t \left[E_k - e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k}\right].$$

Hence,

$$\underset{N_k\to\infty}{p\lim}\left(\widehat{B}_k - B_k\right) = \underset{N_k\to\infty}{p\lim}\left\{\left(X_k^t X_k\right)^- X_k^t \left[E_k - e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k}\right]\right\}.$$

Using the fact that $\underset{N_k\to\infty}{p\lim}\,\widehat{B}_{(k-1)} = B_{(k-1)}$, $\underset{N_k\to\infty}{p\lim}\,\widehat{\Delta}_{(1\ldots k-1)k} = \Delta_{(1\ldots k-1)k}$, we have that

$$\begin{aligned}\underset{N_k\to\infty}{p\lim}\left(\widehat{B}_k - B_k\right) &= \lim_{N_k\to\infty}\left(\frac{1}{N_k}X_k^t X_k\right)^- \underset{N_k\to\infty}{p\lim}\left(\frac{1}{N_k}X_k^t E_k\right)\\ &\quad - \lim_{N_k\to\infty}\left(\frac{1}{N_k}X_k^t X_k\right)^- \underset{N_k\to\infty}{p\lim}\left(\frac{1}{N_k}X_k^t E_{(k-1)}\right)\Delta_{(1\ldots k-1)k} = 0,\end{aligned}$$

in view of Lemma 3 i). Therefore, the proof of part b) of the theorem is now completed.
c) In order to prove this part of the theorem we note that

$$Q(\widehat{B}_1) = (Y_1 - X_1\widehat{B}_1)^t(Y_1 - X_1\widehat{B}_1) = E_1^t E_1,$$

and so in view of Lemma 3 ii) we have that

$$\underset{N_1\to\infty}{p\lim}\,\widetilde{\Delta}_{11} = \underset{N_1\to\infty}{p\lim}\left\{\frac{1}{N_1}Q(\widehat{B}_1)\right\} = \underset{N_1\to\infty}{p\lim}\left(\frac{1}{N_1}E_1^t E_1\right) = \Omega_{11}.$$

Afterwards, based on relation (27), we obtain that

$$\begin{aligned}Y_r - X_r\widehat{B}_r - e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r} &= U_r\left(Y_r - e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\right)\\ &= U_r\left(E_r - e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\right),\end{aligned} \tag{32}$$

with $U_r = I - X_r\left(X_r^t X_r\right)^- X_r^t$. Taking into account that U_r is symmetric and idempotent, we have that

$$Q(\widehat{B}_r, \widehat{\Delta}_{(1\ldots r-1)r}) = \left(E_r - e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\right)^t U_r\left(E_r - e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\right),\ r = 2, \ldots, k.$$

From this relation after some algebra we have

$$\begin{aligned}Q(\widehat{B}_r, \widehat{\Delta}_{(1\ldots r-1)r}) &= E_r^t U_r E_r - E_r^t U_r e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\\ &\quad -\widehat{\Delta}^t_{(1\ldots r-1)r}e^t_{(r-1)}U_r E_r + \widehat{\Delta}^t_{(1\ldots r-1)r}e^t_{(r-1)}U_r e_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\end{aligned}$$

and using that $U_r e_{(r-1)} = U_r Y_{(r-1)} = U_r E_{(r-1)}$, we reach the relation

$$\begin{aligned}Q(\widehat{B}_r, \widehat{\Delta}_{(1\ldots r-1)r}) &= E_r^t U_r E_r - E_r^t U_r E_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}\\ &\quad -\widehat{\Delta}^t_{(1\ldots r-1)r}E^t_{(r-1)}U_r E_r + \widehat{\Delta}^t_{(1\ldots r-1)r}E^t_{(r-1)}U_r E_{(r-1)}\widehat{\Delta}_{(1\ldots r-1)r}.\end{aligned}$$

Based on $\widehat{\Delta}_{(1\dots r-1)r} = \left(E^t_{(r-1)}U_rE_{(r-1)}\right)^- E^t_{(r-1)}U_rE_r$, we obtain that

$$Q(\widehat{B}_r, \widehat{\Delta}_{(1\dots r-1)r}) = E^t_rU_rE_r - E^t_rU_rE_{(r-1)}\left(E^t_{(r-1)}U_rE_{(r-1)}\right)^- E^t_{(r-1)}U_rE_r.$$

Therefore

$$Q(\widehat{B}_r, \widehat{\Delta}_{(1\dots r-1)r}) = E^t_rU_rE_r - \widehat{\Delta}^t_{(1\dots r-1)r}E^t_{(r-1)}U_rE_{(r-1)}\widehat{\Delta}_{(1\dots r-1)r}$$

Afterwards, using that $\underset{N_r\to\infty}{p\lim}\,\widehat{\Delta}_{(1\dots r-1)r} = \Delta_{(1\dots r-1)r}$, for $r = 2, \dots, k$, and applying Lemma 3 ii) and iii) to the matrices $\frac{1}{N_r}E^t_{(r-1)}E_r$, $\frac{1}{N_r}E^t_rE_r$ and $\frac{1}{N_r}E^t_{(r-1)}E_{(r-1)}$, we complete the proof of the theorem. ■

Remark 2. a) We derived in Theorem 5 above the consistent estimators of the parameters B and Ω of the model $Y = XB + E$, $E \sim EC_{N_1\times p}(0, \Sigma\otimes I_{N_1})$, with $Cov\,(E) = \Omega\otimes I_{N_1} = (c\Sigma)\otimes I_{N_1}$. Theorem 5 is only valid for members of $EC_{N_1\times p}(0, \Sigma\otimes I_{N_1})$ which obey the consistency property of Kano (1994), that is for scale mixtures of normal distributions, which include the multivariate normal and multivariate t distributions, as particular cases. Even for these special members of the elliptic family of distributions, the MLE $\widehat{B}$ of B is also a consistent estimator of B. The same is not always true for the MLE $\widehat{\Sigma}$ in view of Theorem 5 c).

It can be easily seen that the MLE $\widehat{\Sigma}$ of Σ is a consistent estimator of the covariance matrix Ω, which means that $\widehat{\Delta}_{11}$ and $\widehat{\Delta}_{rr}$ are consistent estimators of Ω_{11} and $\Omega_{rr} - \Omega_{r(1\dots r-1)}\Omega^{-1}_{(1\dots r-1)(1\dots r-1)}\Omega_{(1\dots r-1)r}$, respectively, for $r = 2, \dots, k$, if the following conditions are fulfilled

$$\lim_{N_1\to\infty}(\lambda_{\max}N_1) = 1 \text{ and } \underset{N_r\to\infty}{p\lim}\left(\frac{\xi_{r,\max}}{h_r}N_r\right) = 1. \tag{33}$$

In the Appendix 1 in Batsidis and Zografos (2005) explicit expressions have been obtained for the quantities $\lambda_{\max}$ and $\xi_{r,\max}$ of specific elliptic distributions. Taking into account this appendix it can be easily seen that (33) are only satisfied for the multivariate normal and t distributions. Hence, in summary, the MLE $\widehat{\Sigma}$ is consistent estimator of the covariance matrix Ω only for the multivariate normal and t distributions. The same has been proved by Sutradhar and Ali (1986) for the case of complete data in the responses. b) If we apply Theorem 5 in the special case of the design matrix $X = 1_{N_1}$, with 1_{N_1} the $N_1\times 1$ unity vector, then we obtain the consistent estimators, in the light of the above remark, for the location parameter, as well as, for the covariance matrix of elliptical distributions studied in Batsidis and Zografos (2005).

4 Test of hypotheses

In the previous section, we obtained the MLEs of the regression parameters B, as well as, of the scale matrix Σ of the model (4), under the assumption that monotone missing data, of the form (5), are available in the response variables. In this section, we will obtain in the context mentioned above, the likelihood ratio test statistic for testing the hypothesis

$$H_0 : C_i B_i = 0_{M_i \times p_i}, \forall\ i = 1, ..., k,$$

with C_i a $M_i \times q$ known coefficient matrix, of rank $M_i \leq q$ and $0_{M_i \times p_i}$, the $M_i \times p_i$ matrix with zero elements. This null hypothesis expresses the existence of M_i linear constraints on the parameters B_i.

In order to derive the likelihood ratio test statistic, we note that the null hypothesis

$$H_0 : C_i B_i = 0, \forall\ i = 1, ..., k,$$

can be equivalently stated in the form

$$H_0 : C_1 B_1 = 0_{M_1 \times p_1} \text{ and } \begin{pmatrix} C_i & 0_{M_i \times p_{(i-1)}} \end{pmatrix} \begin{pmatrix} B_i \\ \Delta_{(1...i-1)i} \end{pmatrix} = 0_{M_i \times p_i}, \forall\ i = 2, ..., k, \quad (34)$$

where $p_{(i-1)} = p_1 + ... + p_{i-1}$ and $p_1 + ... + p_k = p$, for $i \geq 2$.

4.1 Likelihood Ratio Test Statistic

The likelihood function is given by

$$L_Y(B, \Sigma) = L_{Y_1}(\eta_1, \Delta_{11}) L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22}) ... L_{Y_k|Y_{(k-1)}}(\eta_k, \Delta_{(1...k-1)k}, \Delta_{kk}),$$

where $L_{Y_1}(\eta_1, \Delta_{11}), L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22})$ and $L_{Y_k|Y_{(k-1)}}(\eta_k, \Delta_{(1...k-1)k}, \Delta_{kk})$ are defined by (15), (18) and (22) respectively. Using the MLE, obtained in Theorem 1, it can be easily seen that

$$\begin{aligned} \sup_{B,\Sigma} L(B, \Sigma) &= \lambda_{\max}^{-\frac{N_1 p_1}{2}} g_1 \left(\frac{p_1}{\lambda_{\max}} \right) \left| Q(\widehat{B}_1) \right|^{-\frac{N_1}{2}} \xi_{2,\max}^{-\frac{N_2 p_2}{2}} g_2 \left(\frac{p_2}{\xi_{2,\max}} \right) \left| Q(\widehat{B}_2, \widehat{\Delta}_{12}) \right|^{-\frac{N_2}{2}} \\ & \quad \times ... \times \xi_{k,\max}^{-\frac{N_k p_k}{2}} g_k \left(\frac{p_k}{\xi_{k,\max}} \right) \left| Q(\widehat{B}_k, \widehat{\Delta}_{(1...k-1)k}) \right|^{-N_k/2}, \end{aligned} \quad (35)$$

where $Q(\widehat{B}_1), Q(\widehat{B}_2, \widehat{\Delta}_{12})$ and $Q(\widehat{B}_k, \widehat{\Delta}_{(1...k-1)k})$ are defined by (16), (20) and (25) respectively, while $\lambda_{\max}$ and $\xi_{k,\max}$, $k \geq 2$, are defined in Theorem 1.

In order to derive the $\sup_{B,\Sigma} L(B, \Sigma)$, under the null hypothesis $H_0 : C_i B_i = 0,\ \forall\ i = 1, ..., k$, or the equivalent null hypothesis given by (34), following the procedure of Theorem 1, which is based on the conditional likelihood approach, we have, at the beginning, to maximize the quantity $L_{Y_1}(\eta_1, \Delta_{11})$ given by (15), subject to the constraint $C_1 B_1 = 0$. After some algebra, the constraint MLE $\widetilde{B}_1$ of B_1 is given by the following relation

$$\widetilde{B}_1 = \widehat{B}_1 - (X_1^t X_1)^- C_1^t \left[C_1 (X_1^t X_1)^- C_1^t \right]^- C_1 \widehat{B}_1. \quad (36)$$

Following again the procedure of Theorem 1 we easily verify that

$$\widetilde{\Delta}_{11} = \lambda_{\max}(g_1) Q(\widetilde{B}_1). \tag{37}$$

After that, we concentrate our interest on the maximization of $L_{Y_2|Y_{(1)}}(\eta_2, \Delta_{12}, \Delta_{22})$, which is given by (18), subject to $\begin{pmatrix} C_2 & 0_{M_2\times p_1} \end{pmatrix} \begin{pmatrix} B_2 \\ \Delta_{12} \end{pmatrix} = 0_{M_2\times p_2}$. After some algebra we obtain that the MLE of $\begin{pmatrix} B_2 \\ \Delta_{12} \end{pmatrix}$ is

$$\begin{aligned} \begin{pmatrix} \widetilde{B}_2 \\ \widetilde{\Delta}_{12} \end{pmatrix} &= \begin{pmatrix} X_2^t X_2 & X_2^t \widetilde{e}_{(1)} \\ \widetilde{e}_{(1)}^t X_2 & \widetilde{e}_{(1)}^t \widetilde{e}_{(1)} \end{pmatrix}^{-} \begin{pmatrix} X_2^t \\ \widetilde{e}_{(1)}^t \end{pmatrix} Y_2 - \begin{pmatrix} X_2^t X_2 & X_2^t \widetilde{e}_{(1)} \\ \widetilde{e}_{(1)}^t X_2 & \widetilde{e}_{(1)}^t \widetilde{e}_{(1)} \end{pmatrix}^{-} \begin{pmatrix} C_2^t \\ 0^t \end{pmatrix} \\ &\times \left[\begin{pmatrix} C_2 & 0 \end{pmatrix} \begin{pmatrix} X_2^t X_2 & X_2^t \widetilde{e}_{(1)} \\ \widetilde{e}_{(1)}^t X_2 & \widetilde{e}_{(1)}^t \widetilde{e}_{(1)} \end{pmatrix}^{-} \begin{pmatrix} C_2^t \\ 0^t \end{pmatrix} \right]^{-} \begin{pmatrix} C_2 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} X_2^t X_2 & X_2^t \widetilde{e}_{(1)} \\ \widetilde{e}_{(1)}^t X_2 & \widetilde{e}_{(1)}^t \widetilde{e}_{(1)} \end{pmatrix}^{-} \begin{pmatrix} X_2^t \\ \widetilde{e}_{(1)}^t \end{pmatrix} Y_2, \end{aligned}$$

with $\widetilde{e}_{(1)} = Y_{(1)} - \widetilde{\mu}_{(1)}$ and 0 the $M_2 \times p_1$ zero matrix.

Following again Theorem 1, we obtain that

$$\widetilde{\Delta}_{22} = \widetilde{\Sigma}_{2\cdot 1} = \frac{\xi_{2,\max}(g_2)}{h_2(\widetilde{\eta}_1, \widetilde{\Delta}_{11})} Q(\widetilde{B}_2, \widetilde{\Delta}_{12}),$$

where

$$Q(\widetilde{B}_2, \widetilde{\Delta}_{12}) = \left(Y_2 - \begin{pmatrix} X_2 & \widetilde{e}_{(1)} \end{pmatrix} \begin{pmatrix} \widetilde{B}_2 \\ \widetilde{\Delta}_{12} \end{pmatrix} \right)^t \left(Y_2 - \begin{pmatrix} X_2 & \widetilde{e}_{(1)} \end{pmatrix} \begin{pmatrix} \widetilde{B}_2 \\ \widetilde{\Delta}_{12} \end{pmatrix} \right).$$

In a similar manner, the maximization of the quantity $L_{Y_k|Y_{(k-1)}}(\eta_k, \Delta_{(1\ldots k-1)k}, \Delta_{kk})$, defined by (22), subject to the constraints $\begin{pmatrix} C_k & 0_{M_k\times p_{(k-1)}} \end{pmatrix} \begin{pmatrix} B_k \\ \Delta_{(1\ldots k-1)i} \end{pmatrix} = 0_{M_k\times p_k}$, implies that

$$\begin{aligned} \begin{pmatrix} \widetilde{B}_k \\ \widetilde{\Delta}_{(1\ldots k-1)k} \end{pmatrix} &= \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^{-} \begin{pmatrix} X_k^t \\ \widetilde{e}_{(k-1)}^t \end{pmatrix} Y_k - \\ & \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^{-} \begin{pmatrix} C_k^t \\ 0^t \end{pmatrix} \\ &\times \left[\begin{pmatrix} C_k & 0 \end{pmatrix} \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^{-} \begin{pmatrix} C_k^t \\ 0^t \end{pmatrix} \right]^{-} \\ &\times \begin{pmatrix} C_k & 0 \end{pmatrix} \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^{-} \begin{pmatrix} X_k^t \\ \widetilde{e}_{(k-1)}^t \end{pmatrix} Y_k, \end{aligned}$$

with 0 the $M_k \times p_{(k-1)}$ zero matrix and

$$\widetilde{\Delta}_{kk} = \frac{\xi_{k,\max}(g_k)}{h_k} Q(\widetilde{B}_k, \widetilde{\Delta}_{(1...k-1)k}),$$

for $k \geq 3$. Hence, we obtain that

$$\begin{aligned} \sup_{B,\Sigma,H_0:C_iB_i=0} L(B,\Sigma) &= \lambda_{\max}^{-\frac{N_1p_1}{2}} g_1\left(\frac{p_1}{\lambda_{\max}}\right) \left|Q(\widetilde{B}_1)\right|^{-\frac{N_1}{2}} \xi_{2,\max}^{-\frac{N_2p_2}{2}} g_2\left(\frac{p_2}{\xi_{2,\max}}\right) \left|Q(\widetilde{B}_2, \widetilde{\Delta}_{12})\right|^{-\frac{N_2}{2}} \\ &\times ... \times \xi_{k,\max}^{-\frac{N_kp_k}{2}} g_k\left(\frac{p_k}{\xi_{k,\max}}\right) \left|Q(\widetilde{B}_k, \widetilde{\Delta}_{(1...k-1)k})\right|^{-\frac{N_k}{2}}. \end{aligned} \tag{38}$$

The likelihood ratio test statistic for testing the hypothesis $H_0 : C_iB_i = 0, \forall\, i = 1, ..., k$, or its equivalent form (34) is

$$\Lambda = \frac{\sup\limits_{B,\Sigma,H_0:C_iB_i=0} L(B,\Sigma)}{\sup\limits_{B,\Sigma} L(B,\Sigma)},$$

and taking into account (35) and (38), Λ becomes

$$\Lambda = \frac{\sup\limits_{B,\Sigma,H_0:C_iB_i=0} L(B,\Sigma)}{\sup\limits_{B,\Sigma} L(B,\Sigma)} = \frac{\left|Q(\widetilde{B}_1)\right|^{-\frac{N_1}{2}} \left|Q(\widetilde{B}_2, \widetilde{\Delta}_{12})\right|^{-\frac{N_2}{2}} \times ... \times \left|Q(\widetilde{B}_k, \widetilde{\Delta}_{(1...k-1)k})\right|^{-\frac{N_k}{2}}}{\left|Q(\widehat{B}_1)\right|^{-\frac{N_1}{2}} \left|Q(\widehat{B}_2, \widehat{\Delta}_{12})\right|^{-\frac{N_2}{2}} \times ... \times \left|Q(\widehat{B}_k, \widehat{\Delta}_{(1...k-1)k})\right|^{-\frac{N_k}{2}}}. \tag{39}$$

The investigation of the distribution of the test statistic Λ is now in order and it is the subject of the next subsection.

4.2 Distribution of Λ

We observe that the likelihood ratio test statistic, obtained above, coincides with the similar one for testing the same hypothesis $H_0 : C_iB_i = 0, \forall\, i = 1, ..., k$, under the assumption of multivariate normal error (cf. Raats *et al.* (2002)). This point will help us to derive the null distribution of the test statistic Λ. Indeed, by Theorems 8.1.2 and 8.1.3 of Gupta and Varga (1993), we can easily verify that the null distribution of statistic Λ, given in (39), is invariant in the class of elliptical distributions. Hence the null distribution of Λ is the same as the null distribution of Λ in the case of multivariate normal distributed errors in the model (4). This last distribution of Λ has been studied by Raats *et al.* (2002) and Raats (2004). Before we will proceed with the derivation of the distribution of Λ and for the sake of completeness, we give the definition of the said generalized Wilk's distribution.

Definition 1 *Let* $\Lambda_i = \frac{|A_i|}{|A_i+C_i|}$, *with* $A_i \sim W_{d_i}(s_i)$ *and* $C_i \sim W_{d_i}(t_i)$ *independent of* A_i, *and* $W_{d_i}(s_i)$, $i = 1, ..., k$, *denotes the Wishart distribution. Suppose that* Λ_i *are independent and follow Wilk's* $\Lambda-$*distribution* $\Lambda(d_i, t_i, s_i)$, $i = 1, ..., k$. *Then the product* $\prod\limits_{i=1}^{k} \Lambda_i^{a_i}$, *with* $a_1 = 1$, $a_i \in (0,1)$, *for* $i = 2, ..., k$, *follows the generalized Wilk's distribution* $\Lambda_{A,D,T,S}$ *with parameters* $A = [a_1, ..., a_k]$, $D = [d_1, ..., d_k]$, $T = [t_1, ..., t_k]$ *and* $S = [s_1, ..., s_k]$.

In order to produce the null distribution of the test statistic Λ, we can prove, taking into account relation (39), that

$$\Lambda^{2/N_1} = \prod_{i=1}^{k} \Lambda_i^{a_i}, \tag{40}$$

with

$$\Lambda_1 = \frac{\left|Q(\widehat{B}_1)\right|}{\left|Q(\widetilde{B}_1)\right|} \text{ and } \Lambda_i = \frac{\left|Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i})\right|}{\left|Q(\widetilde{B}_i, \widetilde{\Delta}_{(1...i-1)i})\right|}, \text{ for } i = 2, ..., k, \tag{41}$$

where $a_i = \frac{N_i}{N_1}$, for $i = 1, ..., k$.

After some algebra, we can obtain that

$$\begin{aligned} Q(\widetilde{B}_1) &= Q(\widehat{B}_1) + Y_1^t X_1 \left(X_1^t X_1\right)^- C_1^t \left\{C_1 \left(X_1^t X_1\right)^- C_1^t\right\}^- C_1 \left(X_1^t X_1\right)^- X_1^t Y_1 \\ &= Q(\widehat{B}_1) + Y_1^t P_{11} Y_1, \end{aligned} \tag{42}$$

with

$$P_{11} = X_1 \left(X_1^t X_1\right)^- C_1^t \left\{C_1 \left(X_1^t X_1\right)^- C_1^t\right\}^- C_1 \left(X_1^t X_1\right)^- X_1^t, \tag{43}$$

while for $i = 2, ..., k$, we have that

$$Q(\widetilde{B}_i, \widetilde{\Delta}_{(1...i-1)i}) = Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i}) + Y_i^t P_{1i} Y_i, \tag{44}$$

where P_{1i}, $i = 2, ..., k$, is the $N_i \times N_i$ idempotent matrix given by the following relation

$$\begin{aligned} P_{1i} = & \begin{pmatrix} X_i^t \\ \widetilde{e}_{(i-1)}^t \end{pmatrix}^t \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^- \begin{pmatrix} C_i^t \\ 0_{M_i \times p_{(i-1)}}^t \end{pmatrix} \\ & \times \left[\begin{pmatrix} C_i^t \\ 0_{M_i \times p_{(i-1)}}^t \end{pmatrix}^t \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^- \begin{pmatrix} C_k^t \\ 0_{M_k \times p_{(k-1)}}^t \end{pmatrix} \right]^- \\ & \times \begin{pmatrix} C_i^t \\ 0_{M_i \times p_{(i-1)}}^t \end{pmatrix}^t \begin{pmatrix} X_k^t X_k & X_k^t \widetilde{e}_{(k-1)} \\ \widetilde{e}_{(k-1)}^t X_k & \widetilde{e}_{(k-1)}^t \widetilde{e}_{(k-1)} \end{pmatrix}^- \begin{pmatrix} X_i^t \\ \widetilde{e}_{(i-1)}^t \end{pmatrix}. \end{aligned} \tag{45}$$

Hence from (40) using relations (41), (42) and (44) we have that

$$\Lambda^{2/N_1} = \frac{\left|Q(\widehat{B}_1)\right|}{\left|Q(\widehat{B}_1) + Y_1^t P_{11} Y_1\right|} \prod_{i=2}^{k} \left(\frac{\left|Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i})\right|}{\left|Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i}) + Y_i^t P_{1i} Y_i\right|} \right)^{a_i}. \tag{46}$$

Moreover, from (16), we have that

$$Q(\widehat{B}_1) = Y_1^t P_{01} Y_1, \tag{47}$$

with

$$P_{01} = I_{N_1} - X_1 \left(X_1^t X_1\right)^- X_1^t. \tag{48}$$

In this context, taking into account that the quantity $Q(\widehat{B}_i, \widehat{\Delta}_{(1\ldots i-1)i})$ is equal to

$$\left(Y_i - \begin{pmatrix} X_i & e_{(i-1)} \end{pmatrix} \begin{pmatrix} \widehat{B}_i \\ \widehat{\Delta}_{(1\ldots i-1)i} \end{pmatrix}\right)^t \left(Y_i - \begin{pmatrix} X_i & e_{(i-1)} \end{pmatrix} \begin{pmatrix} \widehat{B}_i \\ \widehat{\Delta}_{(1\ldots i-1)i} \end{pmatrix}\right),$$

for $i = 2, ..., k$, we can obtain after some algebra that

$$Q(\widehat{B}_i, \widehat{\Delta}_{(1\ldots i-1)i}) = Y_i^t P_{0i} Y_i, \text{ for } i = 2, ..., k, \tag{49}$$

where P_{0i}, $i = 2, ..., k$, is the $N_i \times N_i$ idempotent matrix given by the following relation

$$P_{0i} = I_{N_i} - \begin{pmatrix} X_i & e_{(i-1)} \end{pmatrix} \begin{pmatrix} X_i^t X_i & X_i^t e_{(i-1)} \\ e_{(i-1)}^t X_i & e_{(i-1)}^t e_{(i-1)} \end{pmatrix}^- \begin{pmatrix} X_i^t \\ e_{(i-1)}^t \end{pmatrix}, \text{ for } i = 2, ..., k. \tag{50}$$

Using these results, it is immediate to see, applying Theorem 7.8.3 of Gupta and Nagar (2000), that under H_0 the random quantity $Q(\widehat{B}_i, \widehat{\Delta}_{(1\ldots i-1)i})$ is described by a Wishart distribution, $W_{p_i}(rank(P_{0i}), \Delta_{ii})$, for $i = 1, ..., k$, while the quantity $Y_i^t P_{1i} Y_i$ can be easily proved that follows a Wishart distribution $W_{p_i}(rank(P_{1i}), \Delta_{ii})$. Moreover, applying Theorem 7.8.5 of Gupta and Nagar (2000), we can easily seen that the $Q(\widehat{B}_i, \widehat{\Delta}_{(1\ldots i-1)i})$ is independent of $Y_i^t P_{1i} Y_i$, because of $P_{0i} P_{1i} = 0_{N_i \times N_i}$, for $i = 1, ..., k$. Hence, the distribution of Λ_i, given by (41), is Wilk's $\Lambda(d_i = p_i, t_i = rank\,(P_{1i})\,, s_i = rank(P_{0i}))$, for $i = 1, ..., k$, with $rank(A)$ being the rank of the matrix A. Thus from Definition 1, we have that under the null hypothesis, the likelihood ratio test statistic Λ of relation (39), follows the generalized Wilk's distribution, $\Lambda_{A,D,T,S}$ with parameters $A = [a_1, ..., a_k]$, $D = [d_1, ..., d_k]$, $S = [s_1, ..., s_k]$ and $T = [t_1, ..., t_k]$, which are given by the following relations

$$a_i = \frac{N_i}{N_1}, d_i = p_i, s_i = rank(P_{0i}) \text{ and } t_i = rank\,(P_{1i})\,. \tag{51}$$

Since we do not have available an analytical expression for the quantiles of the generalized Wilk's distribution, the critical values for testing the hypothesis under examination, are determined by simulation. In order to avoid this procedure, Raats (2004) proved that the generalized Wilk's distribution can be approximated by χ^2-distributions. In particular, motivated from Theorem 3.1 of Raats (2004), we have that a second order approximation of the distribution of

$$V = -2\log\left(\prod_{i=1}^{k} \Lambda_i^{a_i}\right),$$

is

$$P(V \le v) = (1 - w_2) P\left(\chi_f^2 \le vq\right) + w_2 P\left(\chi_{f+4}^2 \le vq\right) + O(N^{-3}), \tag{52}$$

where

$$f = \sum_{i=1}^{k}\sum_{j=1}^{d_i} t_i,$$

$$q = \frac{1}{4f}\sum_{i=1}^{k}\sum_{j=1}^{d_i} \frac{t_i}{a_i}\left(2s_i - 2j + t_i\right), \tag{53}$$

$$w_2 = -\frac{f}{4} + \frac{1}{48q^2}\sum_{i=1}^{k}\sum_{j=1}^{d_i} \frac{t_i}{a_i^2}\left\{3\left(s_i + j + 1\right)\left(s_i + j + t_i + 1\right) + \left(t_i - 2\right)\left(t_i - 1\right)\right\}.$$

Also from Raats (2004) we have that a first order approximation is given by the relation

$$P(V \leq v) = P\left(\chi_f^2 \leq vq\right). \tag{54}$$

The results of Subsections 4.1 and 4.2 are summarized in the next theorem.

Theorem 6 *Under the assumptions of Theorem 1, the likelihood ratio criterion for testing the hypothesis* $H_0 : C_i B_i = 0_{M_i \times p_i}, \forall\, i = 1, ..., k$, *with* C_i *a* $M_i \times q$ *known coefficient matrix, of rank* $M_i \leq q$, *is*

$$\Lambda^{2/N_1} = \frac{\left|Q(\widehat{B}_1)\right|}{\left|Q(\widehat{B}_1) + Y_1^t P_{11} Y_1\right|} \prod_{i=2}^{k} \left(\frac{\left|Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i})\right|}{\left|Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i}) + Y_i^t P_{1i} Y_i)\right|} \right)^{a_i},$$

with $Q(\widehat{B}_1)$ *and* $Q(\widehat{B}_i, \widehat{\Delta}_{(1...i-1)i})$, *for* $i \geq 2$, *are given by (47) and (49), respectively, while* P_{11} *and* P_{1i}, *for* $i \geq 2$, *are given by (42) and (47), respectively. The test statistic follows under the null hypothesis a generalized Wilks distribution* $\Lambda_{A,D,T,S}$ *with parameters* $A = [1, \frac{N_2}{N_1}, ..., \frac{N_k}{N_1}], D = [p_1, ..., p_k]$, $S = [rank(P_{01}), ..., rank(P_{0k})]$ *and* $T = [rank(P_{11}), ..., rank(P_{1k})]$. *Moreover, a second order approximation of the distribution of* $-2\log\Lambda^{2/N_1}$ *is given by relations (52) and (53), while a first order approximation is given by (54).*

Remark 3. If we apply the results of Theorem 6 in the special case of the constant term model obtained from (4), when $X = 1_{N_1}$, with 1_{N_1} the $N_1 \times 1$ unity vector and $B = (B_1, B_2, ..., B_k)$, where each B_i is $1 \times p_i$ dimensional with $p_1 + p_2 + ... + p_k = p$, under the further assumption that $C_i = 1$, we reach the results obtained previously in Krishnamoorty and Pannala (1998).

5 Numerical Example

Sutradhar and Ali (1986) dealt with a multivariate linear regression model under the assumption that the errors have a multivariate t-distribution. This model, which is a direct multidimensional generalization of Zellner's (1976) regression model is used in the area of the stock market analysis. Sutradhar and Ali (1986), in order to illustrate their results, considered a stock market problem relating to four selected firms, having the regression model

$$y_{ij} = a_i + \beta_i m_j + \epsilon_{ij}, i = 1, ..., 4, \ j = 1, ..., 20,$$

where y_{ij} denotes the monthly return on \$100 of capital invested on the i stock during the j month, while m_j denotes the weighted average of these returns during the j month for the aggregate of all stocks trading on the New York Stock Exchange and the error variable was assumed to have a t-distribution. The available data are given in Table 1 of Sutradhar and Ali (1986).

Therefore, if we use the notation of Section 2 we state the regression model

$$Y = XB + E,$$

where Y is a 20×4 observation matrix of responses, X is a known 20×2 model matrix of full column rank, with the elements of the first column equal to 1, $B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ is a 2×4 matrix of regression parameters with unknown values, and the error matrix E is a 20×4 random matrix, which has a matrix variate t distribution $Mt_{20\times4}(0, \Sigma \otimes I_{20})$.

In the sequel and in order to illustrate the main results of this paper, we discard from Table 1 of Sutradhar and Ali (1986) the last six observations on the fourth firm-response variable. That means that we obtain a 2−step monotone missing sample for the response variables with

$$p_1 = 3, \ p_2 = 1, \ N_1 = 20 \text{ and } N_2 = 14.$$

Using the complete data set of Sutradhar and Ali (1986), the estimators $\widehat{B}_c$ of the parameters of the model and the estimator $\widehat{\Sigma}_c$ of Σ are

$$\widehat{B}_c = \begin{pmatrix} -0.2749 & -0.8904 & 0.2159 & -2.0095 \\ 1.1815 & 1.0132 & 0.9513 & 1.1196 \end{pmatrix}$$

and

$$\widehat{\Sigma}_c = \begin{pmatrix} 7.8015 & 6.0204 & -4.0966 & -0.8160 \\ 6.0204 & 12.3543 & -1.1775 & -3.6845 \\ -4.0966 & -1.1775 & 13.6160 & 9.5150 \\ -0.8160 & -3.6845 & 9.5150 & 22.6846 \end{pmatrix}.$$

Using the estimation procedure introduced in Section 3, the respective MLE's are

$$\widehat{B}_M = \begin{pmatrix} -0.2749 & -0.8904 & 0.2159 & -1.0708 \\ 1.1815 & 1.0132 & 0.9513 & 1.1299 \end{pmatrix}$$

and

$$\widehat{\Sigma}_M = \begin{pmatrix} 7.8015 & 6.0204 & -4.0966 & -2.9186 \\ 6.0204 & 12.3543 & -1.1775 & -7.6747 \\ -4.0966 & -1.1775 & 13.6160 & 6.1773 \\ -2.9186 & -7.6747 & 6.1773 & 16.7292 \end{pmatrix}.$$

The MLE's based on partially complete data (the complete data obtained by discarding the additional data on the first $p_1 = 3$ components) are:

$$\widehat{B}_{PC} = \begin{pmatrix} -0.3057 & -0.8884 & 0.2673 & -1.0616 \\ 1.0550 & 0.7955 & 0.7361 & 1.1234 \end{pmatrix}$$

and

$$\widehat{\Sigma}_{PC} = \begin{pmatrix} 9.4591 & 6.2461 & -5.4823 & -2.9323 \\ 6.2461 & 13.8200 & -2.3845 & -9.4512 \\ -5.4823 & -2.3845 & 8.7979 & 3.7541 \\ -2.9323 & -9.4512 & 3.7541 & 17.0437 \end{pmatrix},$$

respectively.

Using measures such as the scale ratio and the likelihood displacement, which was used to measure the influence of dropping observations by Diaz-Garcia *et al.* (2003), we can see that the estimators based on the whole monotone data are closer to the similar one based on complete data than to the estimators based on partially complete data. Thus, in this sense, using the whole monotone data we are able to recover the information lost due to the deletion of the six observations.

In order to illustrate Theorem 6, let us consider in the sequel the following hypothesis testing problem, related to the numerical example of this section. Suppose that we want to test, for instance, according to the notation of Section 4, the null hypothesis

$$H_0 : C_i B_i = 0, \forall i = 1, 2, \text{ against } H_\alpha : \exists i \in \{1, 2\} \text{ such that } C_i B_i \neq 0,$$

where

$$B_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}, B_2 = \begin{pmatrix} a_4 \\ \beta_4 \end{pmatrix}, \text{with } C_i = \begin{pmatrix} -1 & 1 \end{pmatrix}, i = 1, 2.$$

Following the results of Section 4, we obtain that the test statistic of Theorem 6 follows under the null hypothesis a generalized Wilk's distribution $\Lambda_{A,D,T,S}$ with parameters $A = [1, 0.7], D = [3, 1]$, $T = [1, 1]$ and $S = [18, 9]$. Moreover $\Lambda = 0.0016$, $V = -2\log\Lambda^{2/N_1} = 1.2863$, while from relation (53) we have that $f = 4$, $q = 7.7054$ and $w_2 = 0.0158$, with $p-$value $= 0.0456$.

According to the standard procedure for the same hypothesis (cf. for instance Diaz-Garcia *et al.* (2003), Siotani *et al.* (1985, pp. 298-299), Muirhead (1982, pp. 458-460) and references therein) based on partially complete data with $N = 14$, the value of the likelihood ratio test statistic is 0.0032 with $p-$value $= 0.0883$. We observe that the test statistic based on the monotone data, provides more evidence against the null hypothesis than the statistic based on partially complete data. If, in addition, we test the hypothesis by using the whole sample, the value of the test statistic, based on the standard procedure, is 0.0025 with $p-$value$= 0.0487$. This value is closer to the respective value of the test proposed in this paper, than the similar one obtained by using partially complete data method.

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Appendix.

Proof of Proposition 2. Using a mathematical induction argument, we will prove, at the beginning, that the desired results are true for a $k = 2$ step monotone pattern. Then, under the assumption that the conclusions of Proposition 2 holds for $k - 1$ step pattern, we will prove that it is also true for k step monotone pattern.

In the sequel, we will use the following relations, which can be easily obtained,

$$U_r Y_r = \left(I - \frac{1}{N_r} 1_{N_r} 1_{N_r}^t \right) Y_r = Y_r - 1_{N_r} \overline{Y}_r, \tag{55}$$

and

$$Y_r^t U_r Y_r = (U_r Y_r)^t U_r Y_r = \left(Y_r - 1_{N_r}\overline{Y}_r\right)^t \left(Y_r - 1_{N_r}\overline{Y}_r\right) = S_{rr,r} \tag{56}$$

with $U_r = I - \frac{1}{N_r} 1_{N_r} 1_{N_r}^t$, for $r = 1, ..., k$. Moreover,

$$\begin{aligned} e_{(r-1)}^t U_r e_{(r-1)} &= \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right)^t \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right) \\ &= S_{(1\ldots r-1)(1\ldots r-1),r}, \end{aligned} \tag{57}$$

and for $r = 2, ..., k$,

$$\begin{aligned} e_{(r-1)}^t U_r Y_r &= \left(Y_{(r-1)} - 1_{N_r}\widehat{B}_{(r-1)}\right)^t \left(Y_r - 1_{N_r}\overline{Y}_r\right) \\ &= \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right)^t \left(Y_r - 1_{N_r}\overline{Y}_r\right) = S_{(1\ldots r-1)r,r}. \end{aligned} \tag{58}$$

Relation (57) can be easily proved, combining that

$$\begin{aligned} e_{(r-1)}^t e_{(r-1)} &= \left(Y_{(r-1)} - 1_{N_r}\widehat{B}_{(r-1)}\right)^t \left(Y_{(r-1)} - 1_{N_r}\widehat{B}_{(r-1)}\right) \\ &= \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right)^t \left(Y_{(r-1)} - 1_{N_r}\overline{Y}_{(r-1)}\right) \\ &\quad + \left(\overline{Y}_{(r-1)} - \widehat{B}_{(r-1)}\right)^t 1_{N_r}^t 1_{N_r} \left(\overline{Y}_{(r-1)} - \widehat{B}_{(r-1)}\right), \end{aligned}$$

and

$$\begin{aligned} e_{(r-1)}^t 1_{N_r} 1_{N_r}^t e_{(r-1)} &= \left(1_{N_r}^t e_{(r-1)}\right)^t \left(1_{N_r}^t e_{(r-1)}\right) \\ &= N_r^2 \left(\overline{Y}_{(r-1)} - \widehat{B}_{(r-1)}\right)^t \left(\overline{Y}_{(r-1)} - \widehat{B}_{(r-1)}\right). \end{aligned}$$

Now we are ready to present the proof of Proposition 2. For a $k = 2$ step monotone pattern, using Theorem 1 and in view of $X_1 = 1_{N_1}$, we have that

$$\widehat{B}_1 = (X_1^t X_1)^- X_1 Y_1 = (1_{N_1}^t 1_{N_1})^{-1} 1_{N_1}^t Y_1 = \frac{1}{N_1}\sum_{\nu=1}^{N_1} y_{1\nu}^t = \overline{y}_{1,1}^t = \overline{Y}_1.$$

Moreover from Theorem 1 we have that $\widehat{\Delta}_{11} = \lambda_{\max}(g_1)Q(\widehat{B}_1)$, where $Q(\widehat{B}_1) = (Y_1 - X_1\widehat{B}_1)^t(Y_1 - X_1\widehat{B}_1)$. Because of $X_1 = 1_{N_1}$ and $\widehat{B}_1 = \overline{Y}_1$, we obtain that

$$\widehat{\Delta}_{11} = \lambda_{\max}(g_1)(Y_1 - 1_{N_1}\overline{Y}_1)^t(Y_1 - 1_{N_1}\overline{Y}_1) = \lambda_{\max}(g_1)S_{11,1}.$$

Following Remark 1b) we have that the MLE estimator of Δ_{12}, for this special case, is

$$\widehat{\Delta}_{12} = \left[e_{(1)}^t U_2 e_{(1)}\right]^- e_{(1)}^t U_2 Y_2,$$

with $e_{(1)} = Y_{(1)} - \widehat{\mu}_{(1)} = Y_{(1)} - 1_{N_2}\widehat{B}_{(1)} = Y_{(1)} - 1_{N_2}\overline{Y}_1$ and $U_2 = I - \frac{1}{N_2}1_{N_2}1_{N_2}^t$. Taking into account (57) and (58), we have that

$$\widehat{\Delta}_{12} = S_{11,2}^{-1}S_{12,2}. \tag{59}$$

Moreover, using Remark 1 b) we have that

$$\begin{aligned}
\widehat{B}_2 &= \left(1_{N_2}^t 1_{N_2}\right)^- 1_{N_2}^t \left(Y_2 - e_{(1)}\widehat{\Delta}_{12}\right) \\
&= \frac{1}{N_2}1_{N_2}^t Y_2 - \frac{1}{N_2}1_{N_2}^t e_{(1)}\widehat{\Delta}_{12} \\
&= \overline{Y}_2 - (\overline{Y}_{(1)} - \overline{Y}_1)\widehat{\Delta}_{12}
\end{aligned}$$

because it holds that $\frac{1}{N_2}1_{N_2}^t e_{(1)} = \frac{1}{N_2}1_{N_2}^t\left(Y_{(1)} - 1_{N_2}\overline{Y}_1\right) = \overline{Y}_{(1)} - \overline{Y}_1$.

In order to obtain $\widehat{\Delta}_{22}$, we have to compute the quantity $Q(\widehat{B}_2, \widehat{\Delta}_{12})$, which is given, using Theorem 1 and relation (32), by the following relation

$$\begin{aligned}
Q(\widehat{B}_2, \widehat{\Delta}_{12}) &= \left(Y_2 - e_{(1)}\widehat{\Delta}_{12}\right)^t U_2 \left(Y_2 - e_{(1)}\widehat{\Delta}_{12}\right) \\
&= Y_2^t U_2 Y_2 - \widehat{\Delta}_{21}e_{(1)}^t U_2 Y_2 - Y_2^t U_2 e_{(1)}\widehat{\Delta}_{12} + \widehat{\Delta}_{21}e_{(1)}^t U_2 e_{(1)}\widehat{\Delta}_{12}.
\end{aligned}$$

Using (56)-(59) we obtain that

$$\begin{aligned}
Q(\widehat{B}_2, \widehat{\Delta}_{12}) &= S_{22,2} - \widehat{\Delta}_{21}S_{12,2} - S_{21,2}\widehat{\Delta}_{12} + \widehat{\Delta}_{21}S_{11,2}\widehat{\Delta}_{12} \\
&= S_{22,2} - S_{21,2}S_{11,2}^{-1}S_{12,2} = S_{2\cdot1,2}.
\end{aligned}$$

Assuming now that the desired results hold for a $k-1$ step monotone missing pattern, we are going to prove that Proposition 2 remains valid for a k step pattern. The MLE estimator of $\Delta_{(1\ldots k-1)k}$, for $k \geq 3$ is

$$\widehat{\Delta}_{(1\ldots k-1)k} = \left[e_{(k-1)}^t U_k e_{(k-1)}\right]^- e_{(k-1)}^t U_k Y_k,$$

with $e_{(k-1)} = Y_{(k-1)} - \widehat{\mu}_{(k-1)} = Y_{(k-1)} - 1_{N_k}\widehat{B}_{(k-1)}$. In view of relations (57) and (58), we obtain

$$\widehat{\Delta}_{(1\ldots k-1)k} = S_{(1\ldots k-1)(1\ldots k-1),k}^{-1}S_{(1\ldots k-1)k,k}. \tag{60}$$

Moreover, using Remark 1 b)

$$\begin{aligned}
\widehat{B}_k &= \left(1_{N_k}^t 1_{N_k}\right)^{-} 1_{N_k}^t \left(Y_k - e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k}\right) \\
&= \frac{1}{N_k} 1_{N_k}^t Y_k - \frac{1}{N_k} 1_{N_k}^t e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k} \\
&= \overline{Y}_k - (\overline{Y}_{(k-1)} - \widehat{B}_{(k-1)})\widehat{\Delta}_{(1\ldots k-1)k}, \text{ for } k \geq 3,
\end{aligned}$$

because it holds that $1_{N_k}^t e_{(k-1)} = N_k \left(\overline{Y}_{(k-1)} - \widehat{B}_{(k-1)}\right)$.

In order to compute the estimator of $\widehat{\Delta}_{kk}$, we have to compute the quantity $Q(\widehat{B}_k, \widehat{\Delta}_{(1\ldots k-1)k})$, which is given in view of Theorem 1 and (32), by the following relation

$$\begin{aligned}
Q(\widehat{B}_k, \widehat{\Delta}_{(1\ldots k-1)k}) &= \left(Y_k - e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k}\right)^t U_k \left(Y_k - e_{(k-1)}\widehat{\Delta}_{(1\ldots k-1)k}\right) \\
&= Y_k^t U_k Y_k + \widehat{\Delta}_{k(1\ldots k-1)} e_{(k-1)}^t U_k e_{(k-1)} \widehat{\Delta}_{(1\ldots k-1)k} \\
&\quad - \widehat{\Delta}_{k(1\ldots k-1)} e_{(k-1)}^t U_k Y_k - Y_k^t U_k e_{(k-1)} \widehat{\Delta}_{(1\ldots k-1)k}.
\end{aligned}$$

Using relations (56)- (58) and relation (60), we reach the following relation

$$\begin{aligned}
Q(\widehat{B}_k, \widehat{\Delta}_{(1\ldots k-1)k}) &= S_{kk,k} - S_{k(1\ldots k-1),k} S^{-1}_{(1\ldots k-1)(1\ldots k-1),k} S_{(1\ldots k-1)k,k} \\
&= S_{k\cdot(1\ldots k-1),k}.
\end{aligned}$$

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On the solvability of a singular boundary value problem for the equation $f(t, x, x', x'') = 0$

by

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Abstract. In this work we consider boundary value problems of the form

$$f(t, x, x', x'') = 0, \ 0 < t < 1; \ x(0) = 0, \ x'(1) = b, \ b > 0,$$

where the the scalar function $f(t, x, p, q)$ may be singular at $x = 0$. As far as we know, the solvability of the singular boundary value problems of this form has not been treated yet. Here we try to fill in this gap. Examples, illustrating our main result, are included.

Keywords and phrases: Singular boundary value problems, equations unsolved with respect to the second derivative, existence of positive solutions.

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1.INTRODUCTION

In this paper we are dealing with the existence of positive solutions to the boundary value problem

$$f(t, x, x', x'') = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \quad x'(1) = b, \quad b > 0, \tag{1.2}$$

where the scalar function $f(t, x, p, q)$ may be singular at $x = 0$, i.e. f may tend to infinity when x tends to zero on the left and/or on the right hand side. In fact, we need f to be defined at least for

$$(t, x, p, q) \in [0, 1] \times \{D_x \setminus \{0\}\} \times D_p \times D_q,$$

where the sets $D_x, D_p, D_q \subseteq R$ may be bounded. We need also D_x, D_p and D_q to be such that $0 \in D_x$, $0 \in D_q$ and the sets $D_q^+ = D_q \cap (0, +\infty)$, $D_q^- = (-\infty, 0) \cap D_q$ and $\{y \in D_p : y > 0\}$ to be not empty as well as the first derivatives of f to be continuous on a suitable subset of the domain of f.

Results on the solvability of various singular BVPs for ordinary differential equations, whose main nonlinearity does not depend on the highest derivative, can be found, for example, in [1-17] and references therein. The papers [3,15] deal with higher order differential equations. In [3,14,15] the main nonlinearity satisfies Caratheodory conditions, while in [14] a differential equation with impulse effects is considered. The results in [2-4,7,9,13,17] guarantee the existence of positive solutions.

The solvability of various nonsingular BVPs for second-order differential equations, whose main nonlinearity depends on x'', has been investigated in [18-27]. The case where the main nonlinearity of the equations is continuous on the set $[0,1] \times R^3$ is considered in [18-26], while the case where the main nonlinearity is continuous on the set $[0,1] \times R^n \times R^n \times Y$, where $Y \subseteq R^n$, is considered in [27]. The results in these works guarantee the existence of solutions which may change their own sign.

As far as we know, the solvability of singular BVPs for equations of the form (1.1) has not been studied yet. In this paper we want to fill in this gap. In order to establish the existence of positive solutions to the BVP (1.1), (1.2) we proceed as follows. For $\lambda \in [0,1]$ and $n = 1, 2, 3, ...$ we construct a family, say $(\Phi)_\lambda$, of regular BVPs. For example, two-parameter families of BVPs have been used also in [4,5,16]. As in [10,25] suitable "barrier strips" yield a priori bounds independent of λ and n for x, x' and x'', where $x \in C^2[0,1]$ is an eventual solution to the family $(\Phi)_\lambda$. These bounds alow us to apply the topological transversality theorem [28, Chapter I, Theorem 2.6] to prove the solvability of the family $(\Phi)_1$ for each $n = 1, 2, 3...$ Finally, we establish a bound for x_n''' independent of n in appropriate domain so that the Arzela-Askoli theorem yields a solution to the problem (1.1), (1.2) as the limit of a sequence of solutions to the problems $(\Phi)_1$, $n = 1, 2, 3, ...$

2. BASIC HYPOTHESES

In order to obtain our results we make the following three basic hypotheses.

H1. There are positive constants K, Q, P_i, $i = 1, 2, 3, 4$ and a sufficiently small $\varepsilon > 0$ such that

$$P_3 + \varepsilon \le P_1 \le b \le P_2 \le P_4 - \varepsilon,\ P_1 < P_2, (0, P_2 + \varepsilon] \subseteq D_x,\ [P_3, P_4] \subseteq D_p,$$

$$[h_q - \varepsilon, H_q + \varepsilon] \subseteq D_q, \text{ where } h_q = -Q + P_1 - b \text{ and } H_q = Q + P_2 - b,$$

and the following "barrier strips" conditions are satisfied

$$f(t,x,p,q) + Kq \ge 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x^0 \times [P_2, P_4] \times D_q^-, \tag{2.1}$$

$$f(t,x,p,q) + Kq \le 0 \text{ for } (t,x,p,q) \in [0,1] \times D_x^0 \times [P_3, P_1] \times D_q^+, \tag{2.2}$$

$$q\Big(f(t,x,p,q) + Kq\Big) \le 0 \text{ for } (t,x,p,q) \in [0,1] \times (0, P_2 + \varepsilon] \times [P_1, P_2] \times \{D_Q^- \cup D_Q^+\}, \tag{2.3}$$

where $D_x^0 = D_x \setminus \{0\}, D_Q^- = \{z \in D_q : z < -Q\}$ and $D_Q^+ = \{z \in D_q : z > Q\}$.

REMARK. Since $[-Q, Q] \subset [h_q - \varepsilon, H_q + \varepsilon] \subseteq D_q$, the sets D_Q^- and D_Q^+ are not empty.

H2. The functions $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous on the set $[0,1]\times(0,P_2+\varepsilon]\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]$ and there is a constant $K_q>K$ such that

$$f_q(t,x,p,q)\le -K_q \quad \text{for} \quad (t,x,p,q)\in[0,1]\times(0,P_2+\varepsilon]\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon],$$

where K, Q, P_1, P_2, h_q, H_q, and ε are as in **H1**.

H3. The functions $f_t(t,x,p,q)$, $f_x(t,x,p,q)$ and $f_p(t,x,p,q)$ are continuous for $(t,x,p,q)\in[0,1]\times(0,P_2+\varepsilon]\times[P_1,P_2]\times[h_q,H_q]$.

3. AN AUXILIARY RESULT

For $\lambda\in[0,1]$ and $n\in\mathbb{N}$ we construct the family of BVPs

$$\begin{cases} K\Big(x''-(1-\lambda)(x'-b)\Big)=\lambda\left(K\Big(x''-(1-\lambda)(x'-b)\Big)+f\Big(t,x,x',x''-(1-\lambda)(x'-b)\Big)\right)\\ x(0)=\dfrac{1}{n},\quad x'(1)=b, \end{cases} \tag{3.1$_\lambda$}$$

which for $\lambda=1$ includes the BVP (1.1), (1.2) and where the constant $K>0$ is as in **H1**, when it is satisfied. Relatively the following proposition is fulfilled.

LEMMA 3.1. Let **H1** be satisfied and let $x(t)\in C^2[0,1]$ be a solution to the family $(3.1)_\lambda$. Then

$$0<\frac{1}{n}\le x(t)\le P_2+\frac{1}{n},\quad P_1\le x'(t)\le P_2,\quad h_q\le x''(t)\le H_q,\quad \text{for}\quad t\in[0,1],\ n\in\mathbb{N},\ n>1/\varepsilon.$$

Proof. Let the number $n\in\mathbb{N}$, $n>1/\varepsilon$ be fixed and suppose that the set

$$S=\{t\in[0,1]:P_2<x'(t)\le P_4\}$$

is not empty. The continuity of $x'(t)$ and the boundary condition at $t=1$ imply that there is an interval $[\alpha,\beta]\subseteq S$ such that

$$x'(\alpha)>x'(\beta). \tag{3.2}$$

Then there is a $\gamma\in[\alpha,\beta]$ such that

$$x''(\gamma)<0.$$

Without loss of generality, assume that $x(\gamma)\ne 0$. Since $x(t)$ is a solution to $(3.1)_\lambda$, we have

$$\Big(\gamma,x(\gamma),x'(\gamma),x''(\gamma)-(1-\lambda)(x'(\gamma)-b)\Big)\in[0,1]\times D_x^0\times D_p\times D_q.$$

But $x'(\gamma)\in(P_2,P_4]$ and $x''(\gamma)-(1-\lambda)(x'(\gamma)-b)<0$. So,

$$\Big(\gamma,x(\gamma),x'(\gamma),x''(\gamma)-(1-\lambda)(x'(\gamma)-b)\Big)\in[0,1]\times D_x^0\times(P_2,P_4]\times D_q^-$$

and by **H1** we obtain

$$0>K\Big(x''(\gamma)-(1-\lambda)(x'(\gamma)-b)\Big)=$$

$$= \lambda\Big(K\Big(x''(\gamma) - (1-\lambda)(x'(\gamma)-b)\Big) + f\Big(\gamma, x(\gamma), x'(\gamma), x''(\gamma) - (1-\lambda)(x'(\gamma)-b)\Big)\Big) \geq 0,$$

which is impossible. Therefore,

$$x'(t) \leq P_2 \ \text{ for } \ t \in [0,1].$$

Similarly, the assumption that the set

$$S_0 = \{t \in [0,1] : P_3 \leq x'(t) < P_1\}$$

is not empty leads to a contradiction, and therefrom we conclude that

$$0 < P_1 \leq x'(t) \ \text{ for } \ t \in [0,1].$$

But the fact that $x'(t) > 0$ on $[0,1]$ means that $x(t) \geq 1/n$ for $t \in [0,1]$ and for fixed $n \in N$. On the other hand, by the mean value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x(t) - x(0) = x'(\xi)t,$$

from where it follows that

$$x(t) \leq P_2 + 1/n < P_2 + \varepsilon \ \text{ for } \ t \in [0,1].$$

Suppose now that there is $(t_0, \lambda_0) \in [0,1] \times [0,1]$ such that

$$x''(t_0) - (1-\lambda_0)(x'(t_0)-b) < -Q.$$

Then, using the fact that $\Big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)(x'(t_0)-b)\Big) \in [0,1] \times (0, P_2+\varepsilon] \times [P_1, P_2] \times D_Q^-$ and having in mind (2.3), we find that

$$0 > K\Big(x''(t_0) - (1-\lambda_0)(x'(t_0)-b)\Big) =$$

$$= \lambda_0\Big(K\Big(x''(t_0) - (1-\lambda_0)(x'(t_0)-b)\Big) + f\Big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)(x'(t_0)-b)\Big)\Big) \geq 0.$$

The obtained contradiction shows that

$$-Q \leq x''(t) - (1-\lambda)(x'(t)-b) \ \text{ for each } \ (t,\lambda) \in [0,1] \times [0,1].$$

In a similar way, assuming that there exists $(t_1, \lambda_1) \in [0,1] \times [0,1]$ such that

$$x''(t_1) - (1-\lambda_1)(x'(t_1)-b) > Q$$

and using (2.1), we again lead to a contradiction. So, we see that

$$-Q \leq x''(t) - (1-\lambda)(x'(t)-b) \leq Q \ \text{ for } \ (t,\lambda) \in [0,1] \times [0,1]$$

which yields

$$h_q = -Q + P_1 - b \leq x''(t) \leq Q + P_2 - b = H_q \ \text{ for } \ t \in [0,1]. \qquad \square$$

4. AN APPROPRIATE EXTENSION OF THE MAIN NONLINEARITY

In order to prove our main result, it is necessary to extend the function f on the set $[0,1]\times R^3$ in a suitable way. With that end in view, we proceed as follows.

For a fixed $n\in\mathbb{N}$ we construct the functions

$$\varphi=\begin{cases} f(t,(2n)^{-1},p,q), & (t,x,p,q)\in[0,1]\times(-\infty,(2n)^{-1})\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]\\ f(t,x,p,q), & (t,x,p,q)\in[0,1]\times[(2n)^{-1},P_2+\varepsilon)\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]\\ f(t,P_2+\varepsilon,p,q), & (t,x,p,q)\in[0,1]\times(P_2+\varepsilon,\infty)\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon],\end{cases}$$

where h_p, H_p, ε and $P_i, i=1,2$, are the constants of **H1**.

REMARK 2. Observe that any other function considered below, which involves the function φ, depends on this fixed value of $n\in\mathbb{N}$. But, for the sake of simplicity, in the sequel we will omit all n-indexes.

Some properties of the function φ are described by the following two lemmas.

LEMMA 4.1. Let **H2** be satisfied. Then $\varphi(t,x,p,q)$ and its derivative $\varphi_q(t,x,p,q)$ are continuous on $\Omega_x\equiv[0,1]\times R\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]$ and $\varphi_q(t,x,p,q)\le -K_q$ for $(t,x,p,q)\in\Omega_x$.

Proof. Clearly, $\varphi(t,x,p,q)$ and

$$\varphi_q=\begin{cases} f_q(t,(2n)^{-1},p,q), & (t,x,p,q)\in[0,1]\times(-\infty,(2n)^{-1})\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]\\ f_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times[(2n)^{-1},P_2+\varepsilon)\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]\\ f_q(t,P_2+\varepsilon,p,q), & (t,x,p,q)\in[0,1]\times(P_2+\varepsilon,\infty)\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]\end{cases}$$

are continuous on Ω_x. Besides, in view of **H2**,

$$f_q(t,x,p,q)\le -K_q \ \text{ for } \ (t,x,p,q)\in[0,1]\times[(2n)^{-1},P_2+\varepsilon]\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon].$$

In particular, for $(t,p,q)\in[0,1]\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon]$ we have

$$f_q(t,(2n)^{-1},p,q)\le -K_q \quad\text{and}\quad f_q(t,P_2+\varepsilon,p,q)\le -K_q.$$

Consequently

$$\varphi_q(t,x,p,q)\le -K_q \ \text{ for every } \ (t,x,p,q)\in[0,1]\times R\times[P_1-\varepsilon,P_2+\varepsilon]\times[h_q-\varepsilon,H_q+\varepsilon].\square$$

LEMMA 4.2 . Let **H1** be satisfied. Then the function $\varphi(t,x,p,q)$ has the following "barrier strips" properties

$$\varphi(t,x,p,q)+Kq\ge 0 \ \text{ for } \ (t,x,p,q)\in[0,1]\times R\times\{P_2\}\times[h_q-\varepsilon,0), \tag{4.1}$$

$$\varphi(t,x,p,q)+Kq\ge 0 \ \text{ for } \ (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,-Q] \tag{4.2}$$

$$\varphi(t,x,p,q)+Kq\le 0 \ \text{ for } \ (t,x,p,q)\in[0,1]\times R\times\{P_1\}\times[0,H_q+\varepsilon]. \tag{4.3}$$

and

$$\varphi(t,x,p,q)+Kq\le 0 \ \text{ for } \ (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[Q,H_q+\varepsilon]. \tag{4.4}$$

Proof. In particular, by the definition of φ, we see that

$$\varphi(t,x,p,q) = f(t,x,p,q) \text{ for } (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

Now, since $[(2n)^{-1}, P_2+\varepsilon] \subseteq D_x^0$, $[P_2, P_2+\varepsilon] \subseteq [P_2, P_4]$ and $[h_q-\varepsilon, 0) \subseteq D_q^-$, in view of **H1**, we get

$$f(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

Therefore,

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0). \quad (4.5)$$

Next, having in mind **H1** and the fact that $(2n)^{-1} \in D_x^0$, $[P_2, P_2+\varepsilon] \subseteq [P_2, P_4]$ and $[h_q-\varepsilon, 0) \subseteq D_q^-$, we see that

$$f(t,(2n)^{-1},p,q) + Kq \geq 0 \text{ for } (t,p,q) \in [0,1] \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0).$$

But, since the definition of φ implies

$$\varphi(t,x,p,q) = f(t,(2n)^{-1},p,q) \text{ for } (t,x,p,q) \in [0,1] \times (-\infty, (2n)^{-1}) \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0),$$

we conclude that

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times (-\infty, (2n)^{-1}) \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0). \quad (4.6)$$

In a similar way, we obtain

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times (P_2+\varepsilon, \infty) \times [P_2, P_2+\varepsilon] \times [h_q-\varepsilon, 0),$$

which together with (4.5) and (4.6) gives (4.1). Remark that the same reasoning as above yields (4.3).

To prove (4.2), observe first that, by the definition of φ,

$$\varphi(t,x,p,q) = f(t,x,p,q) \text{ for } (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_1, P_2] \times [h_q-\varepsilon, -Q),$$

and then, using (2.1), we obtain

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times [(2n)^{-1}, P_2+\varepsilon] \times [P_1, P_2] \times [h_q-\varepsilon, -Q).$$

Besides, (2.1) implies that

$$f(t,(2n)^{-1},p,q) + Kq \geq 0 \text{ and } f(t,P_2+\varepsilon,p,q) + Kq \geq 0$$

for $(t,p,q) \in [0,1] \times [P_1, P_2] \times [h_q-\varepsilon, -Q)$ and, by the definition of φ, we derive

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times \Big\{(-\infty, (2n)^{-1}) \cup (P_2+\varepsilon, \infty)\Big\} \times [P_1, P_2] \times [h_q-\varepsilon, -Q).$$

Thus, we see that

$$\varphi(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times [h_q-\varepsilon, -Q).$$

Finally, by the same arguments, we conclude that

$$\varphi(t,x,p,q) + Kq \leq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times (Q, H_q+\varepsilon]. \square$$

Now, using the function φ we introduce the function

$$\Phi(t,x,p,q)=\begin{cases}\varphi(t,x,P_1,q), & (t,x,p,q)\in[0,1]\times R\times(-\infty,P_1)\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi(t,x,P_2,q), & (t,x,p,q)\in[0,1]\times R\times(P_2,\infty)\times[h_q-\varepsilon,H_q+\varepsilon],\end{cases}$$

whose properties are describing by the following proposition.

LEMMA 4.3. Let **H2** be satisfied. Then $\Phi(t,x,p,q)$ and its derivative $\Phi_q(t,x,p,q)$ are continuous on $\Omega_p :\equiv [0,1]\times R\times R\times[h_q-\varepsilon,H_q+\varepsilon]$ and $\Phi_q(t,x,p,q)\le -K_q$ for $(t,x,p,q)\in\Omega_p$.
Proof. Clearly, $\Phi(t,x,p,q)$ and

$$\Phi_q(t,x,p,q)=\begin{cases}\varphi_q(t,x,P_2,q), & (t,x,p,q)\in[0,1]\times R\times(P_2,\infty)\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi_q(t,x,p,q), & (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],\\ \varphi_q(t,x,P_1,q), & (t,x,p,q)\in[0,1]\times R\times(-\infty,P_1)\times[h_q-\varepsilon,H_q+\varepsilon]\end{cases}$$

are continuous on Ω_p. Besides, by Lemma 4.1,

$$\varphi_q(t,x,p,q)\le -K_q \text{ for } (t,x,p,q)\in[0,1]\times R\times[P_1,P_2]\times[h_q-\varepsilon,H_q+\varepsilon],$$

and hence it follows that

$$\Phi_q(t,x,p,q)\le -K_q \text{ for } (t,x,p,q)\in\Omega_p.\square$$

In order to extend appropriately the main nonlinearity, we suppose that the condition **H2** is satisfied. We assume also that ψ is a function with the properties

$$\Psi(t,x,p,q) \text{ and } \Psi_q(t,x,p,q) \text{ are continuous on } [0,1]\times R^2\times[H_q+\varepsilon,\infty),$$

$\Psi(t,x,p,H_q+\varepsilon)=\Phi(t,x,p,H_q+\varepsilon)$ and $\Psi_q(t,x,p,H_q+\varepsilon)=\Phi_q(t,x,p,H_q+\varepsilon)$ for $(t,x,p)\in[0,1]\times R^2$ and

$$\Psi_q(t,x,p,q)\le -K_q \text{ for } (t,x,p,q)\in[0,1]\times R^2\times[H_q+\varepsilon,\infty),$$

which is possible because, by Lemma 4.3, $\Phi_q(t,x,p,H_q+\varepsilon)\le -K_q$ for $(t,x,p)\in[0,1]\times R^2$. Finally, suppose that Ψ is a function with the properties

$$\psi(t,x,p,q) \text{ and } \psi_q(t,x,p,q) \text{ are continuous on } [0,1]\times R^2\times(-\infty,h_q-\varepsilon],$$

$\psi(t,x,p,h_q-\varepsilon)=\Phi(t,x,p,h_q-\varepsilon)$ and $\psi_q(t,x,p,h_q-\varepsilon)=\Phi_q(t,x,p,h_q-\varepsilon)$ for $(t,x,p)\in[0,1]\times R^2$ and

$$\psi_q(t,x,p,q)\le -K_q \text{ for } (t,x,p,q)\in[0,1]\times R^2\times(-\infty,h_q-\varepsilon],$$

which is possible since, by Lemma 4.3, $\Phi_q(t,x,p,h_p-\varepsilon)\le -K_q$ for $(t,x,p)\in[0,1]\times R^2$.

Now we are ready to extend the function f to the function defined in $[0,1]\times\mathbb{R}^3$ by

$$\overline{f}_n(t,x,p,q)=\begin{cases}\psi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times(-\infty,h_q-\varepsilon),\\ \Phi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times[h_q-\varepsilon,H_q+\varepsilon],\\ \Psi(t,x,p,q), & (t,x,p,q)\in[0,1]\times R^2\times(H_q+\varepsilon,\infty).\end{cases}$$

The next two lemmas establish some useful properties of the functions $\overline{f}_n$ and its derivative

$$(\overline{f}_n)_q(t,x,p,q) = \begin{cases} \psi_q(t,x,p,q), & (t,x,p,q) \in [0,1] \times R \times R \times (-\infty, h_q - \varepsilon) \\ \Phi_q(t,x,p,q), & (t,x,p,q) \in [0,1] \times R \times R \times [h_q - \varepsilon, H_q + \varepsilon] \\ \Psi_q(t,x,p,q), & (t,x,p,q) \in [0,1] \times R \times R \times (H_q + \varepsilon, \infty) \end{cases}$$

LEMMA 4.4. Let **H2** be satisfied. Then

$$\overline{f}_n(t,x,p,q) \text{ and } (\overline{f}_n)_q(t,x,p,q) \text{ are continuous on } [0,1] \times R^3$$

. and

$$(\overline{f}_n)_q(t,x,p,q) \leq -K_q \text{ for } (t,x,p,q) \in [0,1] \times R^3.$$

Proof. Since the conclusion of this lemma follows by the properties of the functions ψ and Ψ and by Lemma 4.3, the details of the proof are omitted.□

LEMMA 4.5. Let **H1** and **H2** be satisfied. Then the function $\overline{f}_n$ has the following "barrier strip" properties:

$$\overline{f}_n(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_2, P_2+\varepsilon] \times (-\infty, 0),$$

$$\overline{f}_n(t,x,p,q) + Kq \leq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_1-\varepsilon, P_1] \times (0,\infty) \quad (4.7)$$

and

$$q\Big(\overline{f}_n(t,x,p,q) + Kq\Big) \leq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times \Big\{R \setminus [-Q, Q]\Big\}.$$

Proof. The definitions of the functions Φ and $\overline{f}_n$ imply that

$$\overline{f}_n(t,x,p,q) = \varphi(t,x,p,q) \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_1, P_2] \times [h_q - \varepsilon, H_q + \varepsilon]. \quad (4.8)$$

On the other hand, by Lemma 4.2,

$$\varphi(t,x,P_2,q) + Kq \geq 0 \text{ for } (t,x,q) \in [0,1] \times R \times [h_q - \varepsilon, 0).$$

So, from the fact that

$$\overline{f}_n(t,x,p,q) = \Phi(t,x,p,q) = \varphi(t,x,P_2,q), \; p \geq P_2, \; q \in [h_q - \varepsilon, 0)$$

it follows that

$$\overline{f}_n(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_2, P_2+\varepsilon] \times [h_q - \varepsilon, 0). \quad (4.9)$$

Observe that, by Lemma 4.4, for each $(t,x,p,q) \in [0,1] \times R \times [P_2, P_2+\varepsilon] \times (-\infty, 0)$ we have

$$\Big(\overline{f}_n(t,x,p,q) + Kq\Big)_q = (\overline{f}_n)_q(t,x,p,q) + K < (\overline{f}_n)_q(t,x,p,q) + K_q \leq 0,$$

which together with (4.9) yields

$$\overline{f}_n(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times R \times [P_2, P_2+\varepsilon] \times (-\infty, 0).$$

Now, note that the same reasoning as above yields (4.7).

Note also that, in particular, from (4.8) it follows that

$$\overline{f}_n(t,x,p,q) = \varphi(t,x,p,q) \;\text{ for }\; (t,x,p,q) \in [0,1] \times R \times [P_1,P_2] \times (Q, H_q + \varepsilon],$$

from where, according to (4.3), we get

$$\overline{f}_n(t,x,p,q) + Kq \le 0 \;\text{ for }\; (t,x,p,q) \in [0,1] \times R \times [P_1,P_2] \times (Q, H_q + \varepsilon]. \tag{4.10}$$

In view of Lemma 4.4, for each $(t,x,p,q) \in [0,1] \times R \times [P_1,P_2] \times (0,\infty)$ it follows that

$$\Big(\overline{f}_n(t,x,p,q) + Kq \Big)_q = (\overline{f}_n)_q(t,x,p,q) + K < (\overline{f}_n)_q(t,x,p,q) + K_q \le 0.$$

So, by (4.10), we conclude that

$$\overline{f}_n(t,x,p,q) + Kq \le 0 \;\text{ for }\; (t,x,p,q) \in [0,1] \times R \times [P_1,P_2] \times (Q,\infty). \tag{4.11}$$

Finally, observe that the inequality

$$\overline{f}_n(t,x,p,q) + Kq \ge 0 \;\text{ for }\; (t,x,p,q) \in [0,1] \times R \times [P_1,P_2] \times (-\infty,-Q)$$

can be obtained in a similar manner.□

Now, for $\lambda \in [0,1]$ and $n \in \mathbb{N}$, $n > 1/\varepsilon$ consider the family of regular problems

$$\begin{cases} K\Big(x'' - (1-\lambda)(x'-b)\Big) = \lambda\Big(K\Big(x'' - (1-\lambda)(x'-b)\Big) + \overline{f}_n\Big(t,x,x',x'' - (1-\lambda)(x'-b)\Big)\Big) \\ x(0) = \dfrac{1}{n}, \quad x'(1) = b, \end{cases}$$

$$(4.12)_\lambda$$

The following two lemmas establish some useful properties of solutions to the family $(4.12)_\lambda$.

LEMMA 4.6. Let **H1** and **H2** be satisfied and let $x(t) \in C^2[0,1]$ be a solution to the family $(4.12)_\lambda$. Then

$$\frac{1}{n} \le x(t) \le P_2 + \varepsilon, \;\; P_1 \le x'(t) \le P_2, \;\; h_q \le x''(t) \le H_q \;\text{ for }\; t \in [0,1].$$

Proof. Since the conclusions of Lemma 4.5 hold, the proof of this lemma is similar to that of Lemma 3.1.□

The next result is a direct consequence of Lemma 4.6 and the definition of the function $\overline{f}_n$.

LEMMA 4.7. Let **H1** and **H2** be satisfied. Then each $C^2[0,1]$-solution to the family $(4.12)_\lambda$ is also a solution to the family $(3.1)_\lambda$, $\lambda \in [0,1]$.

Proof. Observe that, in view of Lemma 4.6, for each solution $x(t) \in C^2[0,1]$ to $(4.12)_\lambda$ we have

$$\Big(t, x(t), x'(t), x''(t)\Big) \in [0,1] \times [n^{-1}, P_2 + \varepsilon] \times [P_1,P_2] \times [h_q, H_q].$$

On the other hand, the definition of $\overline{f}_n$ implies that

$$\overline{f}_n(t,x,p,q) = f(t,x,p,q) \;\text{ for }\; (t,x,p,q) \in [0,1] \times [n^{-1}, P_2+\varepsilon] \times [P_1,P_2] \times [h_q, H_q]$$

from where the assertion of the lemma follows immediately.□

We conclude this section by proving the following important

LEMMA 4.8. Let **H1** and **H2** be satisfied. Then for each $n \in \mathbb{N},\ n > 1/\varepsilon$ the problem $(3.1)_1$ has at least one solution in $C^2[0,1]$.

Proof. Let n be fixed. Then, using Lemma 4.4, we conclude that the functions

$$F(\lambda,t,x,p,q) := \lambda\,\overline{f}_n(t,x,p,q) + (\lambda-1)Kq \ \text{ and } \ F_q(\lambda,t,x,p,q) = \lambda\,(\overline{f}_n)_q(t,x,p) + (\lambda-1)K$$

are continuous for $(\lambda,t,x,p,q) \in [0,1]^2 \times R^3$ and that

$$F_q(\lambda,t,x,p,q) < 0 \quad \text{for} \quad (\lambda,t,x,p) \in\ [0,1]^2 \times R^3.$$

On the other hand, according to Lemma 4.5, we have

$$\overline{f}_n(t,x,p,H_q) + KH_q \le 0 \ \text{ for } \ (t,x,p) \in [0,1] \times R^2$$

and

$$\overline{f}_n(t,x,p,h_q) + Kh_q \ge 0 \ \text{ for } \ (t,x,p) \in [0,1] \times R^2.$$

So, we see that $F < 0$ for $q = H_q$ and $F > 0$ for $q = h_q$. Thus, there is a unique function $V(\lambda,t,x,p) \in (h_q,H_q)$, which is continuous on the set $[0,1]^2 \times R^2$ and such that the equations

$$q = V(\lambda,t,x,p), \ \ (\lambda,t,x,p) \in [0,1]^2 \times R^2$$

and

$$F(\lambda,t,x,p,q) = 0, \ \ (\lambda,t,x,p,q) \in [0,1]^2 \times R^3$$

are equivalent. This means that for any $\lambda \in [0,1]$ the family $(4.12)_\lambda$ is equivalent to the family of BVPs

$$\begin{cases} x'' - (1-\lambda)(x'-b) = V(\lambda,t,x,x'), \ \ t \in [0,1], \\ x(0) = \dfrac{1}{n}, \quad x'(1) = b. \end{cases} \tag{4.13$_\lambda$}$$

Note that $F(0,t,x,p,0) = 0$ yields

$$V(0,t,x,p) = 0 \ \text{ for } \ (t,x,p) \in [0,1] \times R^2. \tag{4.14}$$

Denote now $C^2_B[0,1] := \{x(t) \in C^2[0,1] : x(0) = 1/n,\ x'(1) = b\}$ and define the maps

$$j : C^2_B[0,1] \to C^1[0,1] \quad \text{by} \quad jx = x,$$

$$L_\lambda : C^2_B[0,1] \to C[0,1] \quad \text{by} \quad L_\lambda x = x'' - (1-\lambda)(x'-b),\ \lambda \in [0,1],$$

and

$$V_\lambda :\ C^1[0,1] \to C[0,1] \quad \text{by} \quad (V_\lambda x)(t) = V(\lambda,t,x(t),x'(t)),\ t \in [0,1],\ \lambda \in [0,1].$$

Let introduce the set

$$U = \Big\{x \in C^2_B[0,1] : \frac{1}{2n} < x < P_2 + \varepsilon,\ P_1 - \varepsilon < x' < P_2 + \varepsilon,\ h_q - \varepsilon < x'' < H_q + \varepsilon\Big\},$$

which is a relatively open set in the convex set $C^2_B[0,1]$ of the Banach space $C^2[0,1]$. Since L_λ, $\lambda \in [0,1]$, is a continuous, linear and one-to-one map of $C^2_B[0,1]$ onto $C[0,1]$, we conclude that L_λ^{-1} exists for each $\lambda \in [0,1]$ and is also a continuous map. In addition, V_λ is a continuous map, while the natural embedding j is a completely continuous map. Therefore, the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \text{ defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_\lambda^{-1} V_\lambda j(x)$$

is a compact map. Moreover, the equations

$$L_\lambda^{-1} V_\lambda j(x) = x \quad \text{and} \quad L_\lambda x = V_\lambda j x$$

are equivalent, i.e. the fixed points of $H_\lambda(x)$ are solutions to the family $(4.13)_\lambda$. Further, obverse that the solutions to $(4.13)_\lambda$ are not elements of ∂U, which means that $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Besides, in view of (4.14), $H_0(x) = n^{-1} + bt$. Since $n^{-1} + bt \in U$, we can apply Theorem 2.2 [28, Chapter I] to conclude that H_0 is an essential map. By the topological transversality Theorem 2.6 [28, Chapter I], $H_1 = L_1^{-1} V_1 j$ is also an essential map. Consequently, the problem $(4.13)_1$ has $C^2[0,1]$-solutions, which are also solutions to the problem $(4.12)_1$. Finally, by Lemma 4.7, the solitions of the problem $(4.12)_1$ are also solutions to the problem $(3.1)_1$. □

5. MAIN RESULT

Using the results of the previous sections, we are ready to prove our main result, which is the following existence

THEOREM 5.1. Let **H1**, **H2** and **H3** be satisfied. Then the problem (1.1), (1.2) has at least one solution $x(t) \in C[0,1] \cap C^2(0,1]$ with the property $x(t) > 0$ on $(0,1]$.

Proof. Consider the sequence $\{x_n(t)\} \subset C^2[0,1]$, where $x_n(t)$, $n \in \mathbb{N}$, $n > 1/\varepsilon$ is a solution to $(3.1)_1$. Note that, by Lemma 4.8, the above sequence exists and, by Lemma 3.1, for $n \in \mathbb{N}$, $n > 1/\varepsilon$ the elements of this sequence satisfy the bounds

$$\frac{1}{n} \le x_n(t) \le P_2 + \varepsilon, P_1 \le x_n'(t) \le P_2, h_q \le x_n''(t) \le H_q, \quad t \in [0,1] \tag{5.1}$$

Therefore, in view of **H2** and **H3**, from the differential equation $(3.1)_1$ we conclude that for $t \in (0,1)$ and h small enough

$$\begin{aligned} &[-f_q(t, x_n(t), x_n'(t), q_{nh}(t))]\,[x_n''(t+h) - x_n''(t)] \\ &= h f_t(T_{1h}) + f_x(T_{2h})[x_n(t+h) - x_n(t)] \\ &+ f_p(T_{3h})[x_n'(t+h) - x_n'(t)] \\ &\to f_t(T_n) + f_x(T_n)x_n'(t) + f_p(T_n)x_n''(t), \quad \text{for} \quad h \to 0, \end{aligned} \tag{5.2}$$

where $T_n \equiv T_n\,(t, x_n(t), x_n'(t), x_n''(t))$ and the points T_{1h}, T_{2h}, T_{3h} and $(t, x_n(t), x_n'(t), q_{nh}(t))$ tend to T_n. Because of (5.1), (5.2) and in view of **H2** and **H3**, it follows that $x_n'''(t)$ exists for every $t \in [0,1]$, is given by the formula

$$x_n'''(t) = \{f_t(T_n) + f_x(T_n)x_n'(t) + f_p(T_n)x_n''(t)\} / \left[-f_q(T_n)\right], \tag{5.3}$$

and is continuous on $[0,1]$.

Next, integrating the inequality $P_1 \le x_n'(t) \le P_2$ from 0 to t with $t \in (0,1]$, we obtain

$$\frac{1}{n} + P_1 t \le x_n(t) \le \frac{1}{n} + P_2 t,\ t \in [0,1]. \tag{5.4}$$

Let the constant $\alpha \in (0,1)$. Then, in view of (5.4)

$$x_n(t) \ge P_1\alpha > 0, \quad t \in [\alpha, 1].$$

According to **H3**, using (5.1) and (5.3) we find that

$$|x_n'''(t)| \le (|f_t| + |f_x||x_n'| + |f_p||x_n''|)/K_q \le C_\alpha, \quad t \in [\alpha, 1],$$

where the constant C_α does not depend of n. Now the Arzela-Askoli theorem guarantees the existence of a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converging uniformly on $C^2[\alpha,1]$ to some function $x \in C^2[\alpha,1]$, which is a solution of the differential equation (1.1) for $t \in [\alpha,1]$. The boundary condition $x'(1) = b$ is obviously satisfied. Thus, for $t \in (0,1]$ there exists a solution $x(t) \in C^2(0,1]$ of the differential equation (1.1), which satisfies the boundary condition $x'(1) = b$. Moreover, according to (5.4), we see that

$$0 < P_1 t \le x(t) \le P_2 t \ \text{ for } t \in (0,1) \tag{5.5}$$

and thus $x \in C[0,1]$ and $x(0) = 0$, which implies that $x(t)$ is a solution to the boundary value problem (1.1), (1.2) for which, in view of (5.5), we have $x(t) > 0$ for every $t \in (0,1]$. □

6. ILLUSTRATIVE EXAMPLES

We conclude our investigation with the following examples, illustrating our main result.

EXAMPLE 6.1. Consider the problem

$$\begin{cases} \exp\Big((t-2)x''\Big) + (x'-5)(x'-10) - 2x'' - \dfrac{x''}{\Big(x(30-x)\Big)^2} = 0, \ 0 < t < 1, \\ \\ x(0) = 0, \ x'(1) = 8. \end{cases}$$

It is easy to check that for $K = 1$, $Q = 15$, $P_1 = 7$, $P_2 = 11$, $P_3 = 6$, $P_4 = 12$ and for a sufficiently small $\varepsilon > 0$ the hypothesis **H1** is satisfied. Hence, the hypothesis **H2** is satisfied for $K_q = 2$. Moreover, $D_x \equiv D_x^0 \equiv (-\infty, 0) \cup (0, 30) \cup (30, \infty)$, $D_p \equiv D_q \equiv R$, $h_q = -16$ and $H_q = 18$. Obviously, the functions

$$f_t(t,x,p,q) = q\exp\left(q(t-2)\right), \quad f_x(t,x,p,q) = \frac{q(60-4x)}{\Big(x(30-x)\Big)^3} \ \text{ and } \ f_p(t,x,p,q) = 2p - 15$$

are continuous for $(t,x,p,q) \in [0,1] \times (0,12] \times [7,11] \times [-16,18]$. Therefore, the hypothesis **H3** is fulfilled and, by Theorem 5.1, the considered problem admits a $C[0,1] \cap C^2(0,1]$-solution.

EXAMPLE 6.2. Consider the problem

$$\begin{cases} \sqrt{225-(x')^2}\sin x' - \dfrac{x''}{\sqrt{400-(x'')^2}\sqrt{x(625-x^2)}} - (x'')^3 - 0.5x'' = 0, \quad 0<t<1, \\ x(0)=0, \quad x'(1)=5. \end{cases}$$

Here $D_p = [-15, 15]$ and $D_q = (-20, 20)$. Since $x(0) = 0$, we will investigate this problem only for $D_x^0 = (0, 25)$. Clearly, the function

$$f(t,x,p,q) = \sqrt{225-p^2}\sin p - \frac{q}{\sqrt{400-q^2}\sqrt{x(625-x^2)}} - q^3 - 0.5q$$

is singular at $x = 0$ and satisfies the hypothesis **H1** for $K = 0.5,\ Q = 10,\ P_1 = 4,\ P_2 = 7,\ P_3 = 3.5,\ P_4 = 7.5$ and a sufficiently small $\varepsilon > 0$. The functions

$$f(t,x,p,q) \quad \text{and} \quad f_q(t,x,p,q) = -\frac{1}{\sqrt{x(625-x^2)}}\frac{400}{\sqrt{(400-q^2)^2}} - 3q^3 - 0.5$$

are continuous on $\Omega \equiv [0,1]\times(0,8+\varepsilon]\times[4-\varepsilon,7+\varepsilon]\times[-11-\varepsilon,12+\varepsilon]$. Besides, $f_q(t,x,p,q) < -0.5-\frac{1}{1500}$ for $(t,x,p,q) \in \Omega$. Thus, **H2** is satisfied for $K_q = 0.5 + \frac{1}{1500}$. Now observe that the functions

$$f_t(t,x,p,q) = 0, \quad f_x(t,x,p,q) = \frac{q}{2\sqrt{400-q^2}}\frac{625-3x^2}{\sqrt{\left(x(625-x^2)\right)^3}}$$

and

$$f_p(t,x,p,q) = \cos p\sqrt{225-p^2}\cos p - \frac{p}{\sqrt{225-p^2}}\sin p$$

are continuous on the set $[0,1] \times (0,8] \times [4,7] \times [-11,12]$. This means that **H3** also is satisfied. Consequently, by Theorem 5.1, the considered problem has a $C[0,1] \cap C^2(0,1]$-solution.

EXAMPLE 6.3. Consider the boundary value problem

$$\begin{cases} f(t,x,x',x'') = 0, \quad 0<t<1, \\ x(0)=0, \quad x'(1)=5, \end{cases}$$

where

$$f(t,x,p,q) = \begin{cases} p+e^{-q}-(2+t)q-6 & \text{for} \quad (t,x,p,q) \in [0,1]\times[0,\infty)\times R^2, \\ -q(x^{-2}+1) & \text{for} \quad (t,x,p,q) \in [0,1]\times(-\infty,0)\times R^2. \end{cases}$$

It is easy to check that for $K = 1,\ Q = 10,\ P_1 = 4,\ P_2 = 7,\ P_3 = 3,\ P_4 = 8$ and a sufficiently small $\varepsilon > 0$ the hypothesis **H1** is satisfied. Note also that the functions

$$f(t,x,p,q) = p + e^{-q} - (2+t)q - 6 \quad \text{and} \quad f_q(t,x,p,q) = -e^{-q} - (2+t)$$

are continuous on the set $\Omega \equiv [0,1]\times(0,8+\varepsilon]\times[4-\varepsilon,7+\varepsilon]\times[-11-\varepsilon,12+\varepsilon]$ and that $f_q(t,x,p,q) < -2$ for $(t,x,p,q) \in \Omega$. So, the hypothesis **H2** is fulfilled for $K_q = 2$. Observe now that

$$f_t(t,x,p,q) = -q, \quad f_x(t,x,p,q) = 0 \quad \text{and} \quad f_p(t,x,p,q) = 1$$

to conclude that **H3** is satisfied. So, by Theorem 5.1, the above problem has a $C[0,1] \cap C^2(0,1]$ -solution.

EXAMPLE 6.4. Consider the problem

$$\begin{cases} f(t,x,x',x'') = 0, \quad 0 < t < 1, \\ x(0) = 0, \quad x'(1) = 5, \end{cases}$$

where

$$f(t,x,p,q) = \begin{cases} \sqrt{225-p^2}\sin p - \dfrac{q^3}{\sqrt{400-q^2}}\sqrt{\dfrac{30-x}{x}} - 0.5q \\ \qquad \text{for} \quad (t,x,p,q) \in [0,1] \times (0,30] \times [-15,15] \times (-20,20), \\ \sqrt{225-p^2}\sin p - \dfrac{q}{\sqrt{400-q^2}}\dfrac{1}{\sqrt{x(x^2-900)}} - q \\ \qquad \text{for} \quad (t,x,p,q) \in [0,1] \times [-30,0) \times [-15,15] \times (-20,20). \end{cases}$$

The function $f(t,x,p,q)$ satisfies the hypothesis **H1** for $K = 0.4$, $Q = 10$, $P_1 = 4$, $P_2 = 7$, $P_3 = 3.5$, $P_4 = 8$ and some sufficiently small $\varepsilon > 0$. Note that the functions

$$f(t,x,p,q) = \sqrt{225-p^2}\,\sin p - \frac{q^3}{\sqrt{400-q^2}}\sqrt{\frac{30-x}{x}} - 0.5q$$

and $f_q(t,x,p,q)$ are continuous on the set $\Omega \equiv [0,1] \times (0, 8+\varepsilon] \times [4-\varepsilon, 7+\varepsilon] \times [-11-\varepsilon, 12+\varepsilon]$ and $f_q(t,x,p,q) \le -0.5$ for $(t,x,p,q) \in \Omega$. So, the hypothesis **H2** is fulfilled for $K_q = 0.5$. Further, observe that the functions

$$f_t(t,x,p,q), \quad f_x(t,x,p,q) \quad \text{and} \quad f_p(t,x,p,q)$$

are continuous on the set $[0,1] \times (0,8] \times [4,7] \times [-11,12]$. Hence, the hypothesis **H3** is also satisfied. Therefore, in view of Theorem 5.1, we see that the above problem has a $C[0,1] \cap C^2(0,1]$-solution.

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Completely Regular Fuzzifying Topological Spaces

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Abstract

Some of the properties of the completely regular fuzzifying topological spaces are investigated. It is shown that a fuzzifying topology τ is completely regular iff it is induced by some fuzzy uniformity or equivalently by some fuzzifying proximity. Also, τ is completely regular iff it is generated by a family of probabilistic pseudometrics.

Key words and phrases: Fuzzifying topology, Fuzzifying proximity, fuzzy uniformity, probabilistic pseudometric.

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Introduction

The fuzzifying topologies were introduced by M. Ying in [15]. A classical topology is a special case of a fuzzifying topology. In a fuzzifying topology τ on a set X, every subset A of X has a degree $\tau(A)$ of belonging to τ, $0 \leq \tau(A) \leq 1$. In [1] we defined the degrees of compactness, of local compactness, Hausdorffnes e.t.c. in a fuzzifying topological space (X, τ). We also gave the notion of convergence of nets and filters and introduced the fuzzifying proximities. Every fuzzifying proximity δ induces a fuzzifying topology τ_δ. In [4] we studied the level classical topologies $\tau^\theta, 0 \leq \theta < 1$, corresponding to a fuzzifying topology τ. In the same paper we studied connectedness and local connectedness in fuzzifying topological spaces as well as the so called sequential fuzzifying topologies. In [3] we introduced the fuzzifying syntopogenous structures. We also proved that every fuzzy uniformity $\mathcal{U}$, as it is defined by Lowen in [8], induces a fuzzifying proximity $\delta_{\mathcal{U}}$ and that , for every fuzzifying proximity δ, there exists at least one fuzzy uniformity $\mathcal{U}$ with $\delta = \delta_{\mathcal{U}}$.
In this paper, we continue with the investigation of fuzzifying topologies. In particular we study the completely regular fuzzifying topologies, i.e those fuzzifying topologies τ for which each level topology τ^θ is completely regular. As in the classical case, we prove that, for a fuzzifying topology τ on X, the following properties are equivalent: (1) τ is completely regular; (2) τ is uniformizable, i.e. it is induced

by some fuzzy uniformity; (3) τ is proximizable, i.e. it is induced by some fuzzifying proximity; (4) τ is generated by a family of so called probabilistic pseudometrics on X. We also give a characterization of completely regular fuzzifying spaces in terms of continuous functions. Many Theorems on classical topologies follow as special cases of results obtained in the paper.

1 Preliminaries

A fuzzifying topology on a set X (see [15]) is a map $\tau : 2^X \to [0,1]$, (where 2^X is the power set of X) satisfying the following conditions:
(FT1) $\tau(X) = \tau(\emptyset) = 1$.
(FT2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$.
(FT3) $\tau(\bigcup A_i) \geq \inf_i \tau(A_i)$.

If τ is a fuzzifying topology on X and $x \in X$, then the τ-neighborhood system of x is the function

$$N_x = N_x^\tau : 2^X \to [0,1], N_x(A) = \sup\{\tau(B) : x \in B \subset A\}.$$

By ([15], Lemma 3.2) we have that $\tau(A) = \inf_{x \in A} N_x(A)$.

Theorem 1.1 *([15], I, Theorem 3.2). If τ is a fuzzifying topology on a set X, then the map $x \to N_x = N_x^\tau$, from X to the fuzzy power set $\mathcal{F}(2^X)$ of 2^X, has the following properties:*
(FN1) $N_x(X) = 1$ and $N_x(A) = 0$ if $x \notin A$.
(FN2) $N_x(A_1 \bigcap A_2) = N_x(A_1) \wedge N_x(A_2)$.
(FN3) $N_x(A) \leq \sup_{x \in D \subset A} \inf_{y \in D} N_y(D)$.
Conversely, if a map $x \to N_x$, from X to $\mathcal{F}(2^X)$, satisfies $(FN1)-(FN3)$, then the map

$$\tau : 2^X \to [0,1], \tau(A) = \inf_{x \in A} N_x(A),$$

is a fuzzifying topology and $N_x = N_x^\tau$ for every $x \in X$.

Let now (X, τ) be a fuzzifying topological space. To every subset A of X corrersponds a fuzzy subset $\bar{A} = \bar{A}^\tau$ of X defined by $\bar{A}(x) = 1 - N_x(A^c)$. A function f, from a fuzzifying topological space (X, τ_1) to another one (Y, τ_2), is said to be continuous at some $x \in X$ (see [2]) if $N_x(f^{-1}(A)) \geq N_{f(x)}(A)$ for every subset A of Y. If f is continuous at every point of X, then it is said $(\tau_1, \tau_2)-$continuous. As it is shown in [2], f is continuous iff $\tau_2(A) \leq \tau_1(f^{-1}(A))$ for every subset A of Y. For $f : X \to Y$ a function and τ a fuzzifying topology on $Y, f^{-1}(\tau)$ is defined to be the weakest fuzzifying topology on X for which f is continuous. By [2], $f^{-1}(\tau)$ is given by the neighborhood structure $N_x(A) = N_{f(x)}(Y \setminus f(A^c))$. If $(\tau_i)_{i \in I}$ is a family of fuzzifying topologies on X, we will denote by $\bigvee_{i \in I} \tau_i$, or by $\sup \tau_i$, the weakest of all fuzzifying topologies on X which are finer than each τ_i. As it is proved in [2], $\bigvee_{i \in I} \tau_i$ is given by the neighborhood structure

$$N_x(A) = \sup\{\inf_{i \in J} N_x^{\tau_i}(A_i) : x \in \bigcap_{i \in J} A_i \subset A\},$$

where the infimum is taken over the family of all finite subsets J of I and all $A_i \subset X, i \in J$. For Y a subset of a fuzzifying topological space $(X, \tau), \tau|_Y$ will be the fuzzifying topology induced on Y by τ, i.e. the fuzzifying topology $f^{-1}(\tau)$ where $f : Y \to X$ is the inclusion map. For a family $(X_i, \tau_i)_{i \in I}$ of fuzzifying topological spaces, the product fuzzifying topology $\tau = \prod \tau_i$ on $X = \prod X_i$ is the weakest fuzzifying topology on X for which each projection $\pi_i : X \to X_i$ is continuous. Thus $\tau = \bigvee_i \pi_i^{-1}(\tau_i)$ and it is given by the neighborhood structure

$$N_x(A) = \sup\{\inf_{i \in J} N_{x_i}(A_i) : x \in \bigcap_{i \in J} \pi_i^{-1}(A_i) \subset A\},$$

where the supremum is taken over the family of all finite subsets J of I and $A_i \subset X_i$, for $i \in J$ (see [2]).
The degree of convergence to an $x \in X$, of a net (x_δ) in a fuzzifying topological space (X, τ), is the number $c(x_\delta \to x) = c^\tau(x_\delta \to x)$ defined by

$$c(x_\delta \to x) = \inf\{1 - N_x(A) : A \subset X, (x_\delta) \quad \text{frequently in } A^c\}.$$

As it is shown in [4], for $A \subset X$ and $x \in X$, we have

$$\bar{A}(x) = \max\{c(x_\delta \to x) : (x_\delta) \ \text{ net in } A\}.$$

The degree of Haudsdorffness of X (see [2]) is defined by

$$T_2(X) = 1 - \sup_{x \neq y} \sup\{c(x_\delta \to x) \wedge c(x_\delta \to y : (x_\delta) \ \text{ net in } X\}.$$

Also, the degree of X being T_1 is defined by

$$T_1(X) = \inf_x \inf_{y \neq x} \sup\{N_x(B) : y \notin B\}.$$

Let now (X, τ) be a fuzzifying topological space. For each $0 \leq \theta < 1$, the family $B_\theta^\tau = \{A \subset X : \tau(A) > \theta\}$ is a base for a classical topology τ^θ on X (see [3]). It is easy to see that a subset B of X is a τ^θ-neighborhood of x iff $N_x(B) > \theta$. By [4], $T_2(X)$ (resp. $T_1(X)$) is the supremum of all $0 \leq \theta < 1$ for which τ^θ is T_2 (resp. T_1). Also, for $\tau = \vee \tau_i$, we have that $\tau^\theta = \sup_i \tau_i^\theta$ (see [3], Theorem 3.5). If $\tau = \prod \tau_i$ is a product fuzzifying topology, then $\tau^\theta = \prod \tau_i^\theta$ (see [3], Theorem 3.5). If Y is a subspace of (X, τ) and $\tau_1 = \tau|Y$, then $\tau_1^\theta = \tau^\theta|Y$. By [3], Theorem 3.10, for a fuzzifying topological space (X, τ), $co(X)$ coincides with the supremum of all $0 < \theta < 1$ for which $\tau^{1-\theta}$ is compact.
Next we will recall the notion of a fuzzifying proximity given in [2]. A fuzzifying proximity on a set X is a map $\delta : 2^X \times 2^X \to [0, 1]$ satisfying the following conditions:
(FP1) $\delta(A, B) = 1$ if the A, B are not disjoint.
(FP2) $\delta(A, B) = \delta(B, A)$.
(FP3) $\delta(\emptyset, B) = 0$.
(FP4) $\delta(A_1 \cup A_2, B) = \delta(A_1, B) \vee \delta(A_2, B)$.
(FP5) $\delta(A, B) = \inf\{\delta(A, D) \vee \delta(D^c, B) : D \subset X\}$.
Every fuzzifying proximity δ induces a fuzzifying topology τ_δ given by the neighborhood structure $N_x(A) = 1 - \delta(x, A^c)$. A fuzzifying proximity δ_1 is said to be finer

than another one δ_2 if $\delta_1(A,B) \leq \delta_2(A,B)$ for all subsets A, B of X. For $f : X \to Y$ a function and δ a fuzzifying proximity on Y, the function

$$f^{-1}(\delta) : 2^X \times 2^X \to [0,1], f^{-1}(\delta)(A,B) = \delta(f(A), f(B)),$$

is a fuzzifying proximity on X (see [2]) and it is the weakest of all fuzzifying proximities δ_1 on X for which f is (δ_1, δ)-proximally continuous, i.e. it satisfies $\delta_1(A,B) \leq \delta(f(A), f(B))$ for all subsets A, B of X. As it is shown in [2], $\tau_{f^{-1}(\delta)} = f^{-1}(\tau_\delta)$.
Let now $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X. We will denote by $\delta = \bigvee_\lambda \delta_\lambda$, or by $\sup \delta_\lambda$, the weakest fuzzifying proximity on X which is finer than each δ_λ. By [2], Theorem 8.10, δ is given by

$$\delta(A,B) = \inf\{\sup_{i,j} \inf_{\lambda\in\Lambda} \delta_\lambda(A_i, B_j)\},$$

where the infimum is taken over all finite collections $(A_i), (B_j)$ of subsets of X with $A = \bigcup A_i, B = \bigcup B_j$. Moreover, $\tau_\delta = \bigvee \tau_{\delta_\lambda}$ (see [2]).
Finally we will recall the definition of a fuzzy uniformity introduced by Lowen in [8]. For a set X, let Ω_X be the collection of all functions $\alpha : X \times X \to [0,1]$ such that $\alpha(x,x) = 1$ for all $x \in X$. For α, β in Ω_X the $\alpha \wedge \beta, \alpha \circ \beta$ and α^{-1} are defined by $\alpha\wedge\beta(x,y) = \alpha(x,y)\wedge\beta(x,y), \alpha\circ\beta(x,y) = \sup_z \beta(x,z)\wedge\alpha(z,y), \alpha^{-1}(x,y) = \alpha(y,x)$. If $\alpha = \alpha^{-1}$, then α is called symmetric. A fuzzy uniformity on X is a non-empty subset $\mathcal{U}$ of Ω_X satisfying the following conditions :
(FU1) If $\alpha, \beta \in \mathcal{U}$, then $\alpha \wedge \beta \in \mathcal{U}$.
(FU2) If $\alpha \in \mathcal{U}$ is such that, for every $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \leq \alpha + \epsilon$, then $\alpha \in \mathcal{U}$.
(FU3) For each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \circ \beta \leq \alpha + \epsilon$.
(FU4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$.
A subset $\mathcal{B}$, of a fuzzy uniformity $\mathcal{U}$, is a base for $\mathcal{U}$ if, for each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$. It is easy to see that, for a subset $\mathcal{B}$ of Ω_X, the following are equivalent :
(1) $\mathcal{B}$ is a base for a fuzzy uniformity on X.
(2) (a) If $\alpha, \beta \in \mathcal{B}$ and $\epsilon > 0$, then there exists $\gamma \in \mathcal{B}$ with $\gamma \leq \alpha \wedge \beta + \epsilon$.

(b) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \circ \beta \leq \alpha + \epsilon$.

(c) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha^{-1} + \epsilon$.

In case (2) is satisfied, the fuzzy uniformity $\mathcal{U}$ for which $\mathcal{B}$ is a base consists of all $\alpha \in \Omega_X$ such that, for each $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$.
By [3], every fuzzy uniformity $\mathcal{U}$ on X induces a fuzzifying proximity $\delta_\mathcal{U}$ defined by

$$\delta_\mathcal{U}(A,B) = \inf_{\alpha\in\mathcal{U}} \sup_{x\in A, y\in B} \alpha(x,y).$$

In case $\mathcal{B}$ is a base for $\mathcal{U}$, then

$$\delta_\mathcal{U}(A,B) = \inf_{\alpha\in\mathcal{B}} \sup_{x\in A, y\in B} \alpha(x,y).$$

Every fuzzy uniformity $\mathcal{U}$ induces a fuzzifying topology $\tau_{\mathcal{U}}$ given by the neighborhood structure

$$N_x(A) = 1 - \delta_{\mathcal{U}}(x, A^c) = 1 - \inf_{\alpha \in \mathcal{U}} \sup_{y \notin A} \alpha(x, y).$$

For every fuzzifying proximity δ there exists at least one compatible fuzzy uniformity, i.e. a fuzzy uniformity $\mathcal{U}$ with $\delta_{\mathcal{U}} = \delta$ (see [3], Theorem 11.4).

2 Probabilistic Pseudometrics

A fuzzy real number is a fuzzy subset u of the real numbers $\mathbf{R}$ which is increasing, left continuous, and such that $\lim_{t \to +\infty} u(t) = 1, \lim_{t \to -\infty} u(t) = 0$. A fuzzy real number u is said to be non-negative if $u(t) = 0$ if $t \le 0$. We will denote by $\mathbf{R}_\phi^+$ the collection of all non-negative fuzzy real numbers. To every real number r corresponds a fuzzy real number $\bar{r}$, where $\bar{r}(t) = 0$ if $t \le r$ and $\bar{r}(t) = 1$ if $t > r$. For $u, v \in \mathbf{R}_\phi^+$, we define $u \preceq v$ iff $v(t) \le u(t)$ for all $t \in \mathbf{R}$. If $\mathcal{A}$ is a non-empty subset of $\mathbf{R}_\phi^+$ and if $u_o \in \mathbf{R}_\phi^+$ is defined by $u_o(t) = \sup_{v \in \mathcal{A}} v(t)$, then u_o is the biggest of all $u \in \mathbf{R}_\phi^+$ with $u \preceq v$ for all $v \in \mathcal{A}$. We will denote u_o by $\inf \mathcal{A}$ or by $\bigwedge \mathcal{A}$. For $u_1, u_2 \in \mathbf{R}_\phi^+$, we define $u = u_1 \oplus u_2 \in \mathbf{R}_\phi^+$ by $u(t) = \sup\{u_1(t_1) \wedge u_2(t_2) : t = t_1 + t_2\}$. Also, for $u \in \mathbf{R}_\phi^+$ and $\lambda > 0$, we define λu by $(\lambda u)(t) = u(\lambda^{-1} t)$. It is easy to see that, for $u \in \mathbf{R}_\phi^+$ and $\lambda > 0$, we have $(\bar{\lambda} \oplus u)(t) = u(t - \lambda)$.

Definition 2.1 *A probabilistic pseudometric on a set X (see [1]) is a mapping $F : X \times X \to \mathbf{R}_\phi^+$ such that, for all x, y, z in X, we have*

$$F(x,x) = \bar{0}, F(x,y) = F(y,x), F(x,z) \preceq F(x,y) \oplus F(y,z).$$

If in addition $F(x,y)(0+) = 0$ when $x \ne y$, then F is called a probabilistic metric.

If r_1, r_2 are non-negative real numbers, then $\overline{r_1} \preceq \overline{r_2}$ iff $r_1 \le r_2$. Also, for $r = |r_1 - r_2|$, we have that

$$\bar{r} = \wedge\{u \in \mathbf{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1} \text{ and } \overline{r_1} \preceq u \oplus \bar{r_2}\}.$$

In fact, let $u_o = \wedge\{u \in \mathbf{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1} \text{ and } \overline{r_1} \preceq u \oplus \overline{r_2}\}$ and assume (say) $r_1 \ge r_2$. Let $u \in \mathcal{R}_\phi^+$ be such that $\bar{r_2} \preceq u \oplus \overline{r_1}, \overline{r_1} \preceq u \oplus \overline{r_2}$. Then $\overline{r_1}(t) \ge (u \oplus \overline{r_2}(t) = u(t - r_2)$ for all t. If $s < r_1$, then $0 = \overline{r_1}(s) \ge u(s - r_2)$ and so $u(r_1 - r_2) = \sup_{s < r_1} u(s - r_2) = 0$ which implies that $\bar{r} \preceq u$. Thus $\bar{r} \preceq u_o$. On the other hand, we have $\bar{r} \oplus \overline{r_2} = \overline{r_1}$ and $\bar{r} \oplus \overline{r_1} = \overline{2r_1 - r_2}$. Since $\overline{r_2} \preceq \overline{2r_1 - r_2}$, it follows that $u_o \preceq \bar{r}$ and hence $\bar{r} = u_o$. Motivated from the above we define the following distance function on $\mathcal{R}_\phi^+$

$$D : \mathcal{R}_\phi^+ \times \mathcal{R}_\phi^+ \longrightarrow \mathcal{R}_\phi^+, \quad D(u_1, u_2) = \wedge\{u \in \mathcal{R}_\phi^+ : u_1 \preceq u_2 \oplus u, u_2 \preceq u \oplus u_1\}.$$

Then D is a probabilistic pseudometric on $\mathcal{R}_\phi^+$. In fact, it is clear that $D(u_1, u_2) = D(u_2, u_1)$. Also, since $u = u \oplus \bar{0}$, when $u \in \mathcal{R}_\phi^+$, we have that $D(u, u) = \bar{0}$. Finally, let $D(u_1, u_2)(t_1) \wedge D(u_2, u_3)(t_2) > \theta > 0$. There are $v_1, v_2 \in \mathcal{R}_\phi^+$ with

$u_1 \preceq v_1 \oplus u_2, u_2 \preceq v_1 \oplus u_1, u_3 \preceq v_2 \oplus u_2, u_2 \preceq v_2 \oplus u_3, v_1(t_1) > \theta, v_2(t_2) > \theta$. Now $u_1 \preceq v_1 \oplus u_2 \preceq v_1 \oplus (v_2 \oplus u_3) = (v_1 \oplus v_2) \oplus u_3$ and $u_3 \preceq v_2 \oplus u_2 \preceq v_2 \oplus (v_1 \oplus u_1) = (v_1 \oplus v_2) \oplus u_1$.Thus $D(u_1, u_3) \preceq v_1 \oplus v_2$ and $D(u_1, u_3)(t_1 + t_2) \geq v_1(t_1) \wedge v_2(t_2) > \theta$. This proves that $D(u_1, u_3) \preceq D(u_1, u_2) \oplus D(u_2, u_3)$ and the claim follows. We will refer to D as the usual probabilistic pseudometric on $\mathcal{R}_\phi^+$.

Let now F be a probabilistic psedometric on X. For $t > 0$, let $u_{F,t}$ be defined on X^2 by $u_{F,t}(x, y) = F(x, y)(t)$. The family $\mathcal{B}_F = \{u_{F,t} : t > 0\}$ is a base for a fuzzy uniformity $\mathcal{U}_F$ on X. Let τ_F be the fuzzifying topology induced by $\mathcal{U}_F$.

In the rest of the paper, we will consider on $\mathcal{R}_\phi^+$ the fuzzifying topology induced by the usual probabilistic pseudometric D.

Theorem 2.2 *A probabilistic pseudometric F, on a fuzzifying topological space (X, τ), is $\tau \times \tau$ continuous iff $\tau_F \leq \tau$.*

Proof : Assume that $\tau_F \leq \tau$ and let G be a subset of $\mathbf{R}_\phi^+$ and $u = F(x_o, y_o)$ with $N_u(G) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{v \notin G} D(v, u)(t) > \theta$. For x, y in X, we have

$$F(x, y) \preceq F(x, x_o) \oplus F(x_o, y_o) \oplus F(y_o, y) = [F(x, x_o) \oplus F(y, y_o)] \oplus F(x_o, y_o).$$

Similarly $F(x_o, y_o) \preceq [F(x, x_o) \oplus F(y, y_o)] \oplus F(x, y)$. Thus

$$D(F(x, y), F(x_o, y_o)) \preceq F(x, x_o) \oplus F(y, y_o).$$

Let

$$A_1 = \{x \in X : F(x, x_o)(t/2) \geq 1 - \theta\}, \quad \text{and} \quad A_2 = \{x \in X : F(y, y_o)(t/2) \geq 1 - \theta\}.$$

If $x \in A_1, y \in A_2$, then

$$D(F(x, y), F(x_o, y_o))(t) \geq F(x, x_o)(t/2) \wedge F(y, y_o)(t/2) \geq 1 - \theta$$

and so $F(x, y) \in G$. Also, $N_{x_o}^\tau(A_1) \geq N_{x_o}^{\tau_F}(A_1) \geq 1 - \sup_{x \notin A_1} F(x, x_o)(t/2) \geq \theta$ and $N_{y_o}^\tau(A_2) \geq \theta$. Therefore,

$$N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{x_o}^\tau(A_1) \wedge N_{y_o}^\tau(A_1) \geq \theta,$$

which proves that $N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{f(x_o, y_o)}(G)$ and so F is $\tau \times \tau$ continuous. Conversely, assume that F is $\tau \times \tau$ continuous and let $N_{x_o}^{\tau_F}(A) > \theta > 0$. Choose $\epsilon > 0$ such that $N_{x_o}^{\tau_F}(A) > \theta + \epsilon$. There exists a $t > 0$ such that $1 - \sup_{x \notin A} F(x, x_o)(t) > \theta + \epsilon$. If

$$Z = \{u \in \mathbf{R}_\phi^+ : D(u, \bar{0})(t) = u(t) > 1 - \theta - \epsilon\},$$

then

$$N_{\bar{0}}(Z) \geq 1 - \sup_{u \notin Z} D(u, \bar{0})(t) \geq \theta + \epsilon > \theta.$$

Since F is $\tau \times \tau$ continuous and $F(x_o, x_o) = \bar{0}$, there exists a subset A_1 of X containing x_o such that $A_1 \times A_1 \subset F^{-1}(Z)$ and $N_{x_o}(A_1) > \theta$. If $x \in A_1$, then $F(x, x_o) \in Z$ and so $F(x, x_o)(t) > 1 - \theta - \epsilon$, which implies that $x \in A$. Thus $A_1 \subset A$ and so $N_{x_o}(A) \geq N_{x_o}^{\tau_F}(A)$ for every subset A of X and every $x_o \in X$. Hence $\tau_F \leq \tau$ and the result follows.

Theorem 2.3 *Let F be a probabilistic pseudometric on a set X, $\tau = \tau_F, (x_\delta)_{\delta\in\Delta}$ a net in X and $x \in X$. Then*

$$c(x_\delta \to x) = \inf_{t>0} \liminf_{\delta} F(x_\delta, x)(t).$$

Proof: Let $d = \inf_{t>0} \liminf_\delta F(x_\delta, x)(t)$ and assume that $d < \theta < 1$. There exists a $t > 0$ such that $\liminf_\delta F(x_\delta, x)(t) < \theta$. Let $A = \{y : F(y, x)(t) > \theta\}$. Then (x_δ) is not eventually in A and so $c(x_\delta \to x) \leq 1 - N_x(A) \leq \sup_{y\notin A} F(y, x)(t) \leq \theta$, which proves that $c(x_\delta \to x) \leq d$. On the other hand, let $c(x_\delta \to x) < r < 1$. There exists a subset B of X such that (x_δ) is not eventually in B and $1 - N_x(B) < r$. Let $s > 0$ be such that $1 - \sup_{y\notin B} F(y, x)(s) > 1 - r$. For each $\delta \in \Delta$, there exists $\delta' \geq \delta$ with $x_{\delta'} \notin B$ and so $F(x_{\delta'}, x)(s) \leq \sup_{y\notin B} F(y, x)(s)$. Thus $d \leq \liminf_\delta F(x_\delta, x)(s) < r$, which proves that $d \leq c(x_\delta \to x)$ and the result follows.

Theorem 2.4 *Let $F_1, F_2, \ldots, F_n$ be probabilistic pseudometrics on X and define F by*

$$F(x, y)(t) = \min_{1\leq k\leq n} F_k(x, y)(t).$$

Then F is a probabilistic pseudometric and $\tau_F = \bigvee_{k+1}^{n} \tau_{F_k}$.

Proof: Using induction on n, it suffices to prove the result in the case of $n = 2$. It follows easily that F is a probabilistic pseudometric. Since $F_1, F_2 \preceq F$, it follows that $\tau_{F_1}, \tau_{F_2} \leq \tau_F$ and so $\tau_o = \tau_{F_1} \vee \tau_{F_2} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{y\notin A} F(y, x)(t) > \theta$. Let $B_i = \{y \in A^c : F_i(y, x)(t) < 1 - \theta\}, i = 1, 2$. Then $A^c = B_1 \cup B_2$ and so $A = A_1 \cap A_2, A_i = B_i^c$. Moreover $N_x^{\tau_{F_i}}(A_i) \geq 1 - \sup_{y\in B_i} F_i(y, x)(t) \geq \theta$ and thus

$$N_x^{\tau_o}(A) \geq N_x^{\tau_o}(A_1) \bigwedge N_x^{\tau_o}(A_2) \geq N_x^{\tau_{F_1}}(A_1) \bigwedge N_x^{\tau_{F_2}}(A_2) \geq \theta$$

This proves that $N_x^{\tau_o}(A) \geq N_x^{\tau_F}(A)$ and the result follows.

For $\mathcal{F}$ a family of probabilistic pseudometrics on a set X, we will denote by $\tau_{\mathcal{F}}$ the supremum of the fuzzifying topologies $\tau_F, F \in \mathcal{F}$, i.e. $\tau_{\mathcal{F}} = \bigvee_{F\in\mathcal{F}} \tau_F$.

Theorem 2.5 *If $\tau = \tau_{\mathcal{F}}$, where $\mathcal{F}$ is a family of probabilistic pseudometrics on a set X, then $T_2(X) = T_1(X) = 1 - \sup_{y\neq x} \inf_{F\in\mathcal{F}} F(x, y)(0+)$.*

Proof: Let $d = 1 - \sup_{y\neq x} \inf_{F\in\mathcal{F}} F(x, y)(0+)$. It is always true that $T_2(X) \leq T_1(X)$. Suppose that $T_1(X) > r > 0$ and let $x \neq y$. Since τ^r is T_1, there exists a τ^r-neighborhood A of x not containing y. Now $N_x(A) > r$ and hence, there are subsets $A_1, \ldots, A_n$ of X and $F_1, \ldots, F_n$ in $\mathcal{F}$ such that $\bigcap A_k \subset A, N_x^{\tau_{F_k}}(A_k > r$. Since y is not in A, there exists a k with $y \notin A_k$. Let $t > 0$ be such that

$$1 - \sup_{z\notin A_k} F_k(z, x)(t) > r \quad \text{and so} \quad \inf_{F\in\mathcal{F}} F(x, y)(t)(0+) \leq F_k(x, y)(t) < 1 - r,$$

which proves that $d \geq r$. Thus $d \geq T_1(X)$. On the other hand, assume that $d > \theta > 0$ and let $x \neq y$. Choose $\epsilon > 0$ such that $d > \theta + \epsilon$. There exists $F \in \mathcal{F}$ with $F(x, y)(0+) < 1 - \theta - \epsilon$ and hence $F(x, y)(t) < 1 - \theta - \epsilon$ for some $t > 0$. Let

$$A = \{z : F(z, x)(t/2) > 1 - \theta - \epsilon\}, \quad B = \{z : F(z, y)(t/2) > 1 - \theta - \epsilon\}.$$

Clearly $x \in A, y \in B$. If $z \in A \cap B$, then

$$F(x,y)(t) \geq F(x,z)(t/2) \wedge F(z,y)(t/2) > 1 - \theta - \epsilon,$$

a contradiction. Thus $A \cap B = \emptyset$. Moreover

$$N_x(A) \geq N_x^{\tau_F}(A) \geq 1 - \sup_{z \notin A} F(x,z)(t/2) \geq \theta + \epsilon > \theta \quad \text{and} \quad N_y(A) > \theta.$$

It follows that $T_2(X) \geq d$ and the proof is complete.

Let us say that a fuzzifying topology τ on a set X is pseudometrizable if there exists a probabilistic pseudometric F on X with $\tau = \tau_F$.

Theorem 2.6 *A fuzzifying topology τ on X is pseudometrizable iff each level topology $\tau^\theta, 0 \leq \theta < 1$, is pseudometrizable.*

Proof: Assume that $\tau = \tau_F$ for some probabilistic pseudometric F and let $0 \leq \theta < 1$. For each positive integer$n$, with $n > 1/(1-\theta)$, let

$$A_n = \{(x,y) \in X^2 : F(x,y)(1/n) > 1 - \theta - 1/n\}.$$

Then $A_{n+1} \subset A_n$ and the family $\mathcal{D} = \{A_n : n \in \mathbf{N}, n > 1/(1-\theta)\}$ is a base for a uniformity $\mathcal{U}$ on X. The topology σ_θ induced by $\mathcal{U}$ is pseudometrizable since $\mathcal{D}$ is countable. Moreover $\sigma_\theta = \tau^\theta$. Indeed, let A be a σ_θ-neighborhood of x. There exists $n \in \mathbf{N}, n > 1/(1-\theta)$, such that $B = \{y : F(x,y)(1/n) > 1 - \theta - 1/n\} \subset A$. Now

$$N_x^\tau(A) \geq N_x^\tau(B) \geq 1 - \sup_{y \notin B} F(x,y)(1/n) \geq \theta + 1/n > \theta$$

and so A is a τ^θ-neighborhood of x. Conversely, assume that A is a τ^θ-neighborhood of x. There exists $\epsilon > 0$ with $N_x(A) > \theta + \epsilon$. Now there exists a positive integer $n > 1/\epsilon$ such that $1 - \sup_{y \notin A} F(x,y)(1/n) > \theta + 1/n$. Hence

$$\{y : F(x,y)(1/n) > 1 - \theta - 1/n\} \subset A,$$

which implies that A is a σ_θ-neighborhood of x. Thus $\tau^\theta = \sigma_\theta$ and therefore each τ^θ is pseudometrizable. Conversely, suppose that each τ^θ is pseudometrizable. By an argument analogous to the one used in the proof of Theorem 3.3 in [4], we show that there exists a family $\{d_\theta : 0 \leq \theta < 1\}$ of pseudometrics on X such that $d_\theta = \sup_{\theta_1 > \theta} d_{\theta_1}$, for each $0 \leq \theta < 1$, and τ^θ coincides with the topology induced by the pseudometric d_θ. Now, for x, y in X, define $F(x,y) : \mathbf{R} \to [0,1]$ by $F(x,y)(t) = 0$ if $t \leq 0$ and $F(x,y)(t) = \sup\{\theta : 0 < \theta \leq 1, d_{1-\theta}(x,y) < t\}$ if $t > 0$. It is clear $F(x,y)$ is increasing and left continuous. For $0 < r < 1$ and $t > d_{1-r}(x,y)$, we have that $F(x,y)(t) \geq r$ and so $\lim_{t\to\infty} F(x,y)(t) = 1$. Also $F(x,x)(t) = 1$ for every x and every $t > 0$. To show that F is a probabilistic pseudometric on X, we must prove that it satisfies the triangle inequality. So, let $F(x,y)(t_1) \wedge F(y,z)(t_2) > \theta > 0$. Then $d_{1-\theta}(x,y) < t_1, d_{1-\theta}(y,z) < t_2$ and so $d_{1-\theta}(x,z) < t_1 + t_2$, which implies that $F(x,z)(t_1 + t_2) \geq \theta$. Thus the triangle inequality is satisfied and F is a probabilistic pseudometric. We will finish the proof by showing that $\tau_F = \tau$. So let $N_x^{\tau_F} > \theta > 0$

and choose $t > 0$ such that $1 - \sup_{y \notin A} F(y,x)(t) > \theta$. If now $d_\theta(x,y) < t$, then $F(x,y)(t) \geq 1 - \theta$ and thus $y \in A$, which proves that A is a $\sigma_\theta = \tau^\theta$ neighborhood of x. Hence $\tau \geq \tau_F$. On the other hand, let B be a τ^θ-neighborhood of x. There exists $\theta_1 > \theta$ such that $N_x(B) > \theta_1$. Now B is a τ_{θ_1}-neighborhood of x and so there exists $t > 0$ such that $\{y : d_{\theta_1}(x,y) < t\} \subset B$. If $F(x,y)(t) > 1 - \theta_1$, then there exists $\alpha > 1 - \theta_1$ such that $d_{1-\alpha}(x,y) < t$ and so $d_{\theta_1}(x,y) < t$. Thus $\{y : F(x,y)(t) > 1 - \theta_1\} \subset B$ and therefore

$$N_x^{\tau_F}(B) \geq 1 - \sup_{y \notin B} F(x,y)(t) \geq \theta_1 > \theta.$$

Thus $\tau_F \geq \tau$ and the result follows.

Theorem 2.7 *Let (X, F) be a probabilistic pseudometric space, $A \subset X$ and $x \in X$. Let*

$$\begin{aligned} \alpha &= \sup\{\inf_{t>0} \liminf_n F(x_n, x)(t) : (x_n) \text{ sequence in } A\} \\ \beta &= \sup\{\liminf_n F(x_n, x)(t_n) : t_n \to 0+, (x_n) \text{ sequence in } A\} \\ \gamma &= \sup\{\liminf_n F(x_n, x)(1/n) : (x_n) \text{ sequence in } A\} \end{aligned}$$

Then $\alpha = \beta = \gamma = \bar{A}(x)$.

Proof: If $(x_n) \subset A$, then

$$\bar{A}(x) \geq c(x_n \to x) = \inf_{t>0} \liminf_n F(x_n, x)(t)$$

and so $\bar{A}(x) \geq \alpha$. Assume that $\beta > \theta > 0$. There exist a sequence (x_n) in A and a sequence (t_n) of positive real numbers, with $t_n \to 0+$, such that $\liminf_n F(x_n, x)(t_n) > \theta$. Let $t > 0$ and choose k such that $t_n < t$ when $n \geq k$. For $m \geq k$ we have $\inf_{n \geq m} F(x_n, x)(t) \geq \inf_{n \geq m} F(x_n, x)(t_n) > \theta$. Thus $\liminf_n F(x_n, x)(t) > \theta$ for each $t > 0$ and so $\alpha \geq \theta$, which proves that $\alpha \geq \beta$. Clearly $\beta \geq \gamma$. Finally, $N_x(A^c) \geq 1 - sup_{y \in A} F(y,x)(1/n)$ and so $\sup_{y \in A} F(y,x)(1/n) \geq 1 - N_x(A^c) = \bar{A}(x) > \bar{A}(x) - 1/n$. Hence, for each $n \in \mathbf{N}$, there exists $x_n \in A$ with $F(x_n, x)(1/n) > \bar{A}(x) - 1/n$. Consequently,

$$\gamma \geq \liminf_n F(x_n, x)(1/n) \geq \liminf_n (\bar{A}(x) - 1/n) = \bar{A}(x)$$

and so $\gamma \geq \bar{A}(x) \geq \alpha \geq \beta \geq \gamma$, which completes the proof.

In view of [4], Theorem 4.14, we have the following

Corollary 2.8 *Every pseudometrizable fuzzifying topological space is* $\mathbf{N}$*-sequential and hence sequential.*

Theorem 2.9 *If (F_n) is a sequence of probabilistic pseudometrics on a set X, then there exists a probabilistic pseudometric F such that $\tau_F = \bigvee_n \tau_{F_n}$.*

Proof: If F is a probabilistic pseudometric on X and if $\bar{F}$ is defined by $\bar{F}(x,y)(t) = F(x,y)(t)$ if $t \leq 1$ and $\bar{F}(x,y)(t) = 1$ if $t > 1$, then $\bar{F}$ is a probabilistic pseudometric on X and $\tau_{\bar{F}} = \tau_F$. Hence, we may assume that $F_n(x,y)(t) = 1$, for all n, if $t > 1$.

For x, y in X, define $F(x,y)$ on $\mathbf{R}$ by $F(x,y)(t) = 0$ if $t \leq 0$ and $F(x,y)(t) = \inf_n[\frac{1}{n}F_n(x,y)](t)$ if $t > 0$. Clearly $F(x,y)$ is increasing and $F(x,y)(t) = 1$ if $t > 1$. Also $F(x,y)$ is left continuous. In fact, let $F(x,y)(t) > \theta > 0$ and choose n such that $(n+1)t > 1$. There exists $0 < s_1 < t$ such that $F_k(x,y)(ks_1) > \theta$ for $k = 1, \ldots, n$. Choose $s_1 < s < t$ such that $(n+1)s > 1$. Now $F_m(x,y)(ms) = 1$ if $m > n$. Thus

$$F(x,y)(s) = \min_{1 \leq k \leq n} [\frac{1}{k}F_k(x,y)](s) > \theta,$$

which proves that $F(x,y)$ is in $\mathbf{R}_\phi^+$. It is clear that $F(x,x) = \bar{0}$. We need to prove that F satisfies the triangle inequality. So assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > \theta > 0$. If m is such that $(m+1)(t_1+t_2) > 1$, then

$$F(x,z)(t_1+t_2) = \min_{1 \leq k \leq m} F_k(x,z)(k(t_1+t_2)).$$

Since

$$F_k(x,z)(k(t_1+t_2)) \geq F_k(x,y)(kt_1) \wedge F_k(y,z)(kt_2) > \theta,$$

it follows that $F(x,z)(t_1+t_2) > \theta$ and so F satisfies the triangle inequality. We will finish the proof by showing that $\tau_F = \bigvee \tau_{F_n}$. To see this, we first observe that $\frac{1}{n}F_n \preceq F$ which implies that $\tau_{F_n} = \tau_{\frac{1}{n}F_n} \leq \tau_F$ and so $\tau_o = \bigvee_n \tau_{\frac{1}{n}F_n} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta$ and choose $\epsilon > 0$ such that $N_x^{\tau_F}(A) > \theta + \epsilon$. Let $t > 0$ be such that $1 - \sup_{y \notin A} F(y,x)(t) > \theta + \epsilon$. If $(m+1)t > 1$, then

$$F(y,z)(t) = \min_{1 \leq k \leq m} F_k(y,z)(kt).$$

Let $A_k = \{y : F_k(y,x)(kt) \geq 1 - \theta - \epsilon\}$. Then

$$N_x^{\tau_o}(A_k) \geq N_x^{\tau_{F_k}}(A_k) \geq 1 - sup_{z \notin A_k} F_k(z,x)(kt) \geq \theta + \epsilon > \theta$$

and $\bigcap_{k=1}^m A_k \subset A$. Hence $N_x^{\tau_o}(A) \geq \min_{1 \leq k \leq m} N_x^{\tau_o}(A_k) > \theta$. This proves that $\tau_F \leq \tau_o$ and the result follows.

Theorem 2.10 *Let $f : X \to Y$ be a function and let F be a probabilistic pseudometric on Y. Then the function*

$$f^{-1}(F) : X^2 \to \mathbf{R}_\phi^{+1}, f^{-1}(F)(x,y) = F(f(x), f(y))$$

is a probabilistic pseudometric on X and $\tau_{f^{-1}(F)} = f^{-1}(\tau_F)$.

Proof: It follows easily that $f^1(F)$ is a probabilistic pseudometric on X. Let $x \in X$ and $B \subset X$. If $D = Y \setminus f(B^c)$, then

$$\begin{aligned} N_x^{\tau_{f^{-1}(F)}}(B) &= \inf_{t>0}[1 - \sup_{y \notin B} F(f(y), f(x))(t)] \\ &= \inf_{t>0}[1 - \sup_{z \in D^c} F(z, f(x))(t)] \\ &= N_{f(x)}^{\tau_F}(D) = N_x^{f^{-1}(\tau_F)}(B), \end{aligned}$$

which clearly completes the proof.

Corollary 2.11 *If F is a probabilistic pseudometric on a set X and $Y \subset X$, then $\tau_F|_Y$ is induced by the probabilistic pseudometric $G = F|_{Y\times Y}, G(x,y) = F(x,y)$.*

Corollary 2.12 *If (X_n, τ_n) is a sequence of pseudometrizable fuzzifying topological spaces, then the cartesian product $(X, \tau) = (\prod X_n, \prod \tau_n)$ is pseudometrizable.*

Proof: Let F_n be be a probabilistic pseudometric pseudometric on X_n inducing τ_n. If $G_n = \pi_n^{-1}(F_n)$, then $\tau_{G_n} = \pi_n^{-1}(\tau_n)$ and so $\tau = \bigvee_n \pi_n^{-1}(\tau_n)$ is pseudometrizable.

3 Level Proximities

Let δ be a fuzzifying proximity on a set X. For each $0 < d \leq 1$, let δ^d be the binary relation on 2^X defined by : $A\delta^d B$ iff $\delta(A,B) \geq d$. It is easy to see that δ^d is a classical proximity on X. We will show that the classical topology σ_d induced by δ^d coincides with τ^{1-d}. In fact, let $x \in A \in \sigma_d$. Then, x is not in the σ_d-closure of A^c, which implies that $x \not\delta^d A^c$, i.e. $\delta(x, A^c) < d$, and so $N_x^\tau(A) = 1 - \delta(x, A^c) > 1 - d$. This proves that $A \in \tau^{1-d}$. Conversely, if $x \in B \in \tau^{1-d}$, then $N_x^\tau(A) > 1 - d$ and thus $\delta(x, A^c) < d$, which implies that x is not in the σ_d-closure of B^c. Hence B^c is σ_d-closed and so B is σ_d-open.

Theorem 3.1 *If δ is a fuzzifying proximity on a set X and $0 < d \leq 1$, then*

$$\delta^d = \bigvee_{0<\theta<d} \delta^\theta.$$

Proof: If $0 < \theta < d$, then δ^θ is coarser than δ^d and so $\delta_o = \bigvee_{0<\theta<d} \delta^\theta$ is coarser than δ^d. On the other hand, let $A\delta_o B$. Since δ_o is finer than δ^θ (for $0 < \theta < d$), we have that $A\delta^\theta B$ and so $\delta(A,B) \geq \theta$, for each $0 < \theta < d$, which implies that $\delta(A,B) \geq d$, i.e. $A\delta^d B$. So δ_o is finer than δ^d and the result follows.

Theorem 3.2 *For a family $\{\gamma_d : 0 < d \leq 1\}$ of classical proximities on a set X the following are equivalent:*

(1) There exists a fuzzifying proximity δ on X such that $\delta^d = \gamma_d$ for all d.

(2) $\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$ for each $0 < d \leq 1$.

Proof: In view of the preceding Theorem, (1) implies (2). Assume now that (2) is satisfied and define δ on $2^X \times 2^X$ by $\delta(A,B) = \sup\{d : A\gamma_d B\}$ (the supremum over the empty family is taken to be zero). It is clear that $\delta(A,B) = 1$ if the A, B are not disjoint. Also $\delta(A,B) = \delta(A,B)$ and $\delta(A,B \geq \delta(A_1,B_1)$ if $A_1 \subset A, B_1 \subset B$. Let now $\delta(A,B) < d < 1$. Then $A \not\gamma_d B$ and so there exists a subset D of X such that $A \not\gamma_d D$ and $D^c \not\gamma_d B$. Since $A \not\gamma_d D$, we have that $\delta(A,D) \leq d$. Similarly $\delta(D^c,B) \leq d$ and so $\inf\{\delta(A,D) \wedge \delta(D^c,B)\} \leq \delta(A,B)$. On the other hand, if $\delta(A,D) \wedge \delta(D^c,B) < \theta < 1$, then $A \subset D^c$ and so $\delta(A,B) \leq \delta(D^c,B) < \theta$. This proves that δ is a fuzzifying proximity on X. We will finish the proof by showing that $\delta^d = \gamma_d$ for all d. Indeed, if $A\gamma_d B$, then $\delta(A,B) \geq d$, i.e. $A\delta^d B$. On the other hand, let $A\delta^d B$ and let $(A_i), (B_j)$ be finite families of subsets of X with $A = \cup_i, B = \cup B_j$. Since $\delta(A, B) = \bigvee_{i,j} \delta(A_i, B_j) \geq d$, there exists a pair (i, j) such that $\delta(A_i, B_j) \geq d$. If now $0 < \theta < d$, then there exists $r > \theta$ with $A_i \gamma_r B_j$ and so $A_i \gamma_\theta B_j$. This proves that $A \gamma_d B$ since $\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$. This completes the proof.

Theorem 3.3 *Let $(X, \delta_1), (Y\delta_2)$ be fuzzifying proximity spaces and let $f : X \to Y$ be a function. Then f is proximally continuous iff $f : (X, \delta_1^d) \to (Y, \delta_2^d)$ is proximally continuous for each $0 < d \leq 1$.*

Proof: It follows immediately from the definitions.

Theorem 3.4 *Let $(X_\lambda, \delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximity spaces and let $(X, \delta) = (\prod X_\lambda, \prod \delta_\lambda)$ be the product fuzzifying proximity space. Then $\delta^d = \prod \delta_\lambda^d$ for all $0 < d \leq 1$.*

Proof: Since each projection $\pi_\lambda : (X, \delta^d) \to (X_\lambda, \delta_\lambda^d)$ is proximally continuous, it follows that δ^d is finer than $\sigma = \prod \delta_\lambda^d$. On the other hand, let $A\sigma B$. We need to show that $\delta(A, B) \geq d$. In fact, let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \cup A_i, B = \cup B_j$. Since $A\sigma B$ and $\sigma = \bigvee_\lambda \pi_\lambda^{-1}(\delta_\lambda^d)$, there exists a pair (i, j) such that $A_i \pi_\lambda^{-1}(\delta^d) B_j$, i.e. $\delta_\lambda(\pi_\lambda(A_i), \pi_\lambda(B_j)) \geq d$. In view of Theorem 8.9 in [2], we conclude that $\delta(A, B) \geq d$. Hence $\sigma = \delta^d$ and the proof is complete.

We have the following easily established

Theorem 3.5 *Let (Y, δ) be a fuzzifying proximity space and let $f : X \to Y$. Then $f^{-1}(\delta)^d = f^{-1}(\delta^d)$ for each $0 < d \leq 1$.*

Theorem 3.6 *Let $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X and $\delta = \vee_\lambda \delta_\lambda$. Then $\delta^d = \bigvee_\lambda \delta_\lambda^d$ for each $0 < d \leq 1$.*

Proof: Let $\sigma = \bigvee_\lambda \delta_\lambda^d$. Since δ is finer than each δ_λ, it follows that δ^d is finer than each δ_λ^d and so δ^d is finer than σ. On the other hand, let $A\sigma B$ and let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \cup A_i, B = \cup B_j$. There exists a pair (i, j) such that $A_i \sigma B_j$. Since σ is finer than each δ_λ^d, we have that $A_i \delta_\lambda^d B_j$, i.e. $\delta_\lambda(A_i, B_j) \geq d$. In view of Theorem 8.10 in [2], we get that $\delta(A, B) \geq d$, i.e. $A\delta^d B$. So σ is finer than δ^d and the proof is complete.

4 Completely Regular Fuzzifying Spaces

Definition 4.1 *A fuzzifying topological space (X, τ) is called completely regular if each of the classical level topologies $\tau^d, 0 \leq d < 1$ is completely regular.*

Definition 4.2 *A fuzzifying proximity δ on a set X is said to be compatible with a fuzzifying topology τ if τ coincides with the fuzzifying topology τ_δ induced by δ.*

We have the following easily established

Theorem 4.3 *Subspaces and cartesian products of completely regular fuzzifying spaces are completely regular.*

Theorem 4.4 *Let* (X,τ) *be a completely regular fuzzifying topological space and define* $\delta = \delta(\tau) : 2^X \times 2^X \to [0,1]$ *by*

$$\delta(A,B) = 1 - \sup\{d : 0 \leq d < 1, \exists f : (X,\tau^d) \to [0,1] \ \textit{continuous}\ f(A)=0, f(B)=1\}.$$

Then: (1) δ *is a fuzzifying proximity on* X *compatible with* τ.

(2) If δ_1 *is any fuzzifying proximity on* X *compatible with* τ*, then* δ *is finer than* δ_1.

Proof: It is easy to see that δ satisfies $(FP1), (FP2), (FP3)$ and $(FP5)$. We will prove that δ satisfies $(FP4)$. Let

$$\alpha = \inf\{\delta A, D) \vee \delta(D^c, B) : D \subset X\}.$$

If $\delta(A,D) \vee \delta(D^c,B) < \theta$, then $A \subset D^c$ and so $\delta(A,B) \leq \delta(D^c,B) < \theta$, which proves that $\delta(A,B) \leq \alpha$. On the other hand, assume that $\delta(A,B) < r < 1$. There exist a $d, 1-r < d < 1$, and $f : X \to [0,1]$ τ^d-continuous such that $f(A) = 0, f(B) = 1$. Let $D = \{x \in X : 1/2 \leq f(x) \leq 1\}$ and define $h_1, h_2 : [0,1] \to [0,1], h_1(t) = 2t, h_2(t) = 0$ if $0 \leq t \leq 1/2$ and $h_1(t) = 1, h_2(t) = 2t-1$ if $1/2 < t \leq 1$. If $g_i = h_i \circ f, i = 1,2$, then $g_1(A) = 0, g_1(D) = 1, g_2(D^c) = 0, g_2(B) = 1$. Thus $\delta(A,D) \leq 1-d < r, \delta(D^c,B) < r$, which proves that $\alpha \leq \delta(A,B)$. Hence δ is a fuzzifying proximity on X. We need to show that $\tau = \tau_\delta$. So, let $\tau(A) > \theta > 0$. Since τ^θ is completely regular, given $x \in A$, there exists $f_x : X \to [0,1], \tau^\theta$-continuous, $f_x(x) = 0, f_x(A^c) = 1$. Thus $\delta(x,A^c) \leq 1-\theta$ and so $N_x^{\tau_\delta}(A) = 1 - \delta(x,A^c) \geq \theta$. It follows that $\tau_\delta(A) = \inf_{x\in A} N_x^{\tau_\delta}(A) \geq \theta$, which proves that $\tau_\delta \geq \tau$. On the other hand, assume that $\tau_\delta(A) > r > 0$. If $x \in A$, then $\delta(x,A^c) = 1 - N_x^{\tau_\delta}(A) < 1-r$, and therefore there exists a $d, 0 < 1-d < 1-r$ and $f : X \to [0,1]$ τ^d-continuous such that $f(x) = 0, f(A^c) = 1$. The set $G = \{y : f(y) < 1/2\}$ is in τ^d and $x \in G \subset A$. Thus

$$N_x^\tau(A) \geq N_x^\tau(G) \geq d > r.$$

This proves that $\tau(A) \geq r$ and so $\tau \geq \tau_\delta$, which completes the proof of (1).
Let δ_1 be a fuzzifying proximity on X compatible with τ and let A, B be subsets of X with $\delta_1(A,B) < \theta < 1$. If $d = 1-\theta$, then δ_1^θ is compatible with τ^d. Since $A \not\delta_1^\theta B$, there exists (by [11], Remarks 3.15) an $f : X \to [0,1]$ τ_d-continuous, with $f(A) = 0, f(B) = 1$, and so $\delta(A,B) \leq 1-d = \theta$, which proves that $\delta(A,B) \leq \delta_1(A,B)$ and therefore δ is finer than δ_1. This completes the proof.

Theorem 4.5 *For a fuzzifying topological space* (X,τ)*, the following are equivalent:*
(1) (X,τ) *is completely regular.*
(2) There exists a fuzzifying proximity δ *on* X *compatible with* τ.
(3) (X,τ) *is fuzzy uniformizable, i.e. there exists a fuzzy uniformity* $\mathcal{U}$ *on* X *such that* τ *coincides with the fuzzifying topology* $\tau_{\mathcal{U}}$ *induced by* $\mathcal{U}$.

Proof: By [3], (2) is equivalent (3). Also (1) implies (2) in view of the preceding Theorem. Assume now that $\tau = \tau_\delta$ for some fuzzifying proximity δ. For each $0 < d \leq 1, \delta^d$ is a classical proximity compatible with τ^{1-d} and so τ^{1-d} is completely regular. This completes the proof.

Theorem 4.6 *Every pseudometrizable fuzzy topological space (X,τ) is completely regular.*

Proof: If τ is pseudometrizable, then each $\tau^d, 0 \le d < 1$, is pseudometrizable and hence τ^d is completely regular.

Theorem 4.7 *For a fuzzifying topological space (X,τ), the following are equivalent:*
(1) (X,τ) is completely regular.
(2) If $\mathcal{F} = \mathcal{F}_\tau = cp(X)$ is the family of all probabilistic pseudometrics on X which are $\tau \times \tau$ continuous as functions from X^2 to $\mathbf{R}_\phi^+$, then $\tau = \tau_{\mathcal{F}_\tau}$.
(3) There exists a family $\mathcal{F}$ of probabilistic pseudometrics on X such that $\tau = \tau_{\mathcal{F}}$.

Proof: (1) $\Rightarrow$ (2). For each $F \in \mathcal{F}_\tau$, we have that $\tau_F \le \tau$ (by Theorem 2.2) and so $\tau_{\mathcal{F}_\tau} \le \tau$. Let now $A \subset X$ and $x_o \in X$ with $N_{x_o}^\tau(A) > \theta > 0$. Since τ^θ is completely regular, there exists a τ^θ-continuous function f from X to [0,1] such that $f(x_o) = 0, f(A^c) = 1$. For $x, y \in X$, define $F(x,y)$ on $\mathbf{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } \ t \le 0 \\ 1-\theta & \text{if } \ |f(x)-f(y)| \ge t > 0 \\ 1 & \text{if } \ |f(x)-f(y)| < t \end{cases}$$

Clearly $F(x,y) = F(y,x) \in \mathbf{R}_\phi^+$ and $F(x,x) = \bar{0}$. We will prove that F satisfies the triangle inequality. So, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > F(x,z)(t_1+t_2)$. Then, $t_1, t_2 > 0, F(x,z)(t_1+t_2) = 1-\theta, F(x,y)(t_1) = F(y,z)(t_2) = 1$. Thus $t_1 > |f(x)-f(y)|, t_2 > |f(y)-f(z)|$ and hence $|f(x)-f(z)| < t_1+t_2$, which implies that $F(x,z)(t_1+t_2) = 1$, a contradiction. So F is a probabilistic pseudometric on X. Next we show that F is $\tau \times \tau$ continuous, or equivalently that $\tau_F \le \tau$. So assume that $N_x^{\tau_F}(B) > r > 0$. Let $\theta_1 > r$ be such that $N_x^{\tau_F}(B) > \theta_1$. Choose $t > 0$ such that $1 - \sup_{y \notin B} F(x,y)(t) > \theta_1$ and so $F(x,y)(t) = 1-\theta$ and $|f(x)-f(y)| \ge t$ if $y \notin B$. Thus $\{y : |f(x)-f(y)| < t\} \subset B$. This shows that B is a τ^θ-neighborhood of x. As $r < \theta$, B is a τ^r-neighborhood of x, i.e. $N_x^\tau(B) > r$ and so $\tau_F \le \tau$. Finally if $y \notin A$, then $|f(y)-f(x_o)| = 1$ and so $F(y,x_o)(1/2) = 1-\theta$, which implies that

$$N_{x_o}^{\tau_{\mathcal{F}}}(A) \ge N_{x_o}^{\tau_F}(A) \ge 1 - \sup_{y \notin A} F(y,x_o)(1/2) \ge \theta.$$

This shows that $N_{x_o}^{\tau_{\mathcal{F}}} \ge N_{x_o}^\tau$ and so $\tau \le \tau_{\mathcal{F}}$, which completes the proof of the implication (1) $\Rightarrow$ (2).
(3) $\Rightarrow$ (1) Assume that $\tau = \tau_{\mathcal{F}}$ for some family $\mathcal{F}$ of probabilistic pseudometrics on X. For each $F \in \mathcal{F}, \tau_F$ is completely regular and so $\tau_{\mathcal{F}}$ is completely regular since $\tau_{\mathcal{F}}^d = \bigvee_{F \in \mathcal{F}} \tau_F^d$ for each $0 \le d < 1$. Hence the result follows.

We will denote by $[0,1]_\phi$ the subspace of $\mathbf{R}_\phi^+$ consisting of all $u \in \mathbf{R}_\phi^+$ with $u(t) = 1$ if $t > 1$.

Theorem 4.8 *A fuzzifying topological space (X,τ) is completely regular iff the following condition is satisfied: If $N_{x_o}(A) > \theta > 0$, then there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1-\theta$ if $y \notin A$ and $0 < t < 1$.*

Proof: Assume that (X,τ) is completely regular and let $N_{x_o}(A) > \theta > 0$. Since τ^θ is completely regular, there exists $h : (X,\tau^\theta) \to [0,1]$ continuous, $h(x_o) = 0, h(y) = 1$ if $y \notin A$. For x, y in X, define $F(x,y)$ on $\mathbf{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } \ t \le 0 \\ 1-\theta & \text{if } \ |h(x) - h(x_o)| \ge t > 0 \\ 1 & \text{if } \ |h(x) - h(x_o)| < t \end{cases}$$

Clearly $F(x,y) \in [0,1]_\phi$. Also $F(x,z) \preceq F(x,y) \oplus F(y,z)$. In fact, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > r > F(x,z)(t_1+t_2)$. Then $t_1, t_2 > 0, F(x,y)(t_1) = F(y,z)(t_2) = 1$. Now $|h(x)-h(y)| < t_1, |h(y)-h(z)| < t_2$ and so $|h(x)-h(z)| < t_1+t_2$ which implies that $F(x,z)(t_1+t_2) = 1$, a contradiction. So F is a probabilistic pseudometric. Moreover F is $\tau \times \tau$ continuous, or equivalently $\tau_F \le \tau$. In fact, let $N_x^{\tau_F}(B) > r > 0$. There exists a $t > 0$ such that $1 - \sup_{z \notin B} F(z,x)(t) > r$. If $z \notin B$, then $F(z,x)(t) < 1 - r < 1$ and so $F(z,x)(t) = 1 - \theta < 1 - r$, i.e. $r < \theta$, and $|h(z) - h(x)| \ge t$. Hence

$$M = \{z : |h(z) - h(x)| < t\} \subset B.$$

The set M is a τ^θ-neighborhood of x and hence a τ^r-neighborhood, i.e. $N_x^\tau(B) > r$. Thus $\tau \ge \tau_F$. Finally, define $f : X \to [0,1]_\phi, f(y) = F(y,x_o)$. Then f is τ-continuous, $f(x_o) = \bar{0}$. For $y \notin A$ and $0 < t < 1$, we have that $f(y)(t) = F(y,x_o)(t) = 1 - \theta$ (since $|h(x) - h(x_o| = 1 \ge t)$. Conversely, assume that the condition is satisfied and let $\mathcal{F}$ be the family of all $\tau \times \tau$ continuous pseudometrics on X. Then $\tau_\mathcal{F} \le \tau$. Let $N_{x_o}^\tau(A) > \theta$. There exists a $\theta_1 > \theta$ such that $N_{x_o}^\tau(A) > \theta_1$. By our hypothesis, there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta_1$ if $y \notin A$ and $0 < t < 1$. Define $F(x,y) = D(f(x), f(y))$. Then F is $\tau \times \tau$ continuous and

$$\begin{aligned} N_{x_o}^{\tau_\mathcal{F}}(A) &\ge N_{x_o}^{\tau_F}(A) \ge 1 - \sup_{y \notin A} F(x_o, y)(1) \\ &= 1 - \sup_{y \notin A} D(\bar{0}, f(y))(1) \\ &= 1 - \sup_{y \notin A} f(y)(1) \ge \theta_1 > \theta. \end{aligned}$$

Thus $N_{x_o}^{\tau_\mathcal{F}}(A) \ge N_{x_o}^\tau(A)$, for every subset A of X and so $\tau \le \tau_\mathcal{F}$. Therefore, $\tau = \tau_\mathcal{F}$ and so τ is completely regular.

For a fuzzifying topological space X, we will denote by $C(X,[0,1]_\phi)$ the family of all continuous functions from X to $[0,1]_\phi$.

Theorem 4.9 *A fuzzifying topological space (X,τ) is completely regular iff τ coincides with the weakest of all fuzzifying topologies τ_1 on X for which each $f \in C(X,[0,1]_\phi)$ is continuous.*

Proof: Assume that (X,τ) is completely regular and let τ_1 be the weakest of all fuzzifying topologies on X for which each $f \in C(X,[0,1]_\phi)$ is continuous. Clearly $\tau_1 \le \tau$. On the other hand, let τ_2 be a fuzzifying topology on X for which each $f \in C(X,[0,1]_\phi)$ is continuous. Let $N_x^\tau(A) > \theta > 0$. In view of the preceding Theorem, there exists an $f \in C(X,[0,1]_\phi)$ such that $f(x) = \bar{0}, f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$. Let

$$G = \{u \in \mathbf{R}_\phi^+ : D(f(x), u)(1/2) = u(1/2) > 1 - \theta\}.$$

Then

$$N_{\bar{0}}(G) \geq 1 - \sup_{u \notin G} D(f(x), u)(1/2) \geq \theta.$$

Since f is τ_2-continuous, we have that $N_x^{\tau_2}(f^{-1}(G)) \geq \theta$. But $f^{-1}(G) \subset A$ since, for $y \notin A$, we have that $f(y)(1/2) = 1 - \theta$. Thus $N_x^{\tau_2}(A) \geq \theta$. This proves that $N_x^{\tau_2}(A) \geq N_x^{\tau}(A)$, for every subset A of X and so $\tau_2 \geq \tau$. This clearly proves that $\tau_1 = \tau$. Conversely, assume that $\tau_1 = \tau$. If σ is the usual fuzzifying topology of $\mathbf{R}_\phi^+$, then

$$\tau = \tau_1 = \bigvee_{f \in C(X,[0,1]_\phi)} f^{-1}(\sigma).$$

Since σ is completely regular, each $f^{-1}(\sigma)$ is completely regular and so τ is completely regular. This completes the proof.

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Oscillations of First Order Linear Delay Dynamic Equations

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Abstract

Consider the first order linear delay dynamic equation of the form

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0. \qquad (E)$$

New oscillation criteria are established which contain well-known criteria for delay differential and difference equations as special cases. Illustrative examples are given to show that the results obtained essentially improve known oscillation results for Eq. (E).

1 Introduction and Preliminaries

A time scale T is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced in 1988 by Hilger [7] in his Ph.D. Thesis in order to unify continuous and discrete analysis. Several authors have expounded on various aspect of this new theory. See [1, 2, 8] and the references cited therein.

First we give a short review on the time scales calculus extracted from [1]. For any $t \in T$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in T : s > t\} \qquad \text{and} \qquad \rho(t) := \sup\{s \in T : s < t\},$$

respectively. While the graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

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A point $t \in T$ is said to be right-dense if $t < \sup T$ and $\sigma(t) = t$, left-dense if $t > \inf T$ and $\rho(t) = t$. Also, t is said to be right-scattered if $\sigma(t) > t$, left-scattered $t > \rho(t)$. A function $f : T \to R$ is called rd-continuous if it is continuous at right-dense points in T and its left-sided limits exist (finite) at left-dense points in T.

For a function $f : T \to R$, if there exists a number $\alpha \in R$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with $|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$, for all $s \in U$, then f is Δ-differentiable at t, and we call α the derivative of f at t and denote it by $f^{\Delta}(t)$.

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if t is right-scattered. When t is a right-dense point, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

Assume $f : T \to R$ is Δ-differentiable at $t \in T$, then f is continuous at t. Furthermore, we assume that $g : T \to R$ is Δ-differentiable. The following formulae are useful:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t), \qquad (f(t)g(t))^{\Delta} = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t).$$

A function F with $F^{\Delta} = f$ is called an antiderivative of f, and then we define

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in T$. It is well known that rd-continuous functions possess antiderivatives.

Note that if $T = R$, we have $\sigma(t) = t, \;\; \mu(t) = 0, \;\; f^{\Delta}(t) = f'(t)$ and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt, \tag{1}$$

and if $T = N$, we have $\sigma(t) = t + 1, \;\; \mu(t) = 1, \;\; f^{\Delta} = \Delta f$ and

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \tag{2}$$

while the integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t). \tag{3}$$

If f is rd-continuous, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{4}$$

Intermediate Value Theorem ([8]). *The continuous mapping $f : [r,s] \to R$, is assumed to fulfill the condition: $f(r) < 0 < f(s)$, $r, s \in T$. Then there is a $\delta \in [r,s)$ with $f(\delta)f(\sigma(\delta)) \leq 0$.*

In recent years, there has been an increasing interest in the oscillation of solutions of some dynamic equations. See [1, 2, 14] and the references cited therein. However, few papers only ([3, 13, 18, 19]) deal with delay dynamic equations even in the case of first order linear equations. In this paper, we are concerned with the oscillatory behavior of the first order linear delay dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{E}$$

where $t \in T$, $\tau : T \to T$ is nondecreasing, $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$ and $p : T \to R$ is a nonnegative rd-continuous function.

If $x : T \to R$ is defined and $\Delta-$differentiable for $t \in T$ and satisfies Eq. (E) for $t \in T$, then $x(t)$ is called a solution of Eq. (E). A solution x has a generalized zero at t in case $x(t) = 0$. We say that x has a generalized zero on $[a,b]$ in case $x(t)x(\sigma(t)) < 0$ or $x(t) = 0$ for some $t \in [a,b)$, where a, $b \in T$ and $a \leq b$. (x has a generalized zero at b, in case $x(\rho(b))x(b) < 0$ or $x(b) = 0$). A nontrivial solution of Eq. (E) is said to be oscillatory on $[t_x, \infty)$ if it has infinitely many generalized zeros when $t \geq t_x$. Finally, Eq. (E) is called oscillatory if all its solutions are oscillatory.

We list the following well-known oscillation criteria for the equation (E) in special cases of T. If $T = R$, then Eq. (E) reduces to the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0. \tag{E_R}$$

In 1972, Ladas et al. [11] proved that Eq. (E_R) is oscillatory if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 1, \qquad (S_R)$$

while, in 1979, Ladas [10] and in 1982 Koplatadze and Canturijia [9], proved that the same conclusion holds if

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \qquad (I_R)$$

Similarly, in the case that $T = N$, Eq. (E) reduces to the first order delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \qquad k \in N, \quad n > k \geq 1. \qquad (E_N)$$

In 1989, Erbe and Zhang [5] proved that Eq. (E_N) is oscillatory provided that

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i > 1. \qquad (S_N)$$

In the same year, Ladas et.al [12] presented the following condition

$$\liminf_{n\to\infty}\sum_{i=n-k}^{n-1} p_i > (\frac{k}{k+1})^{k+1} \qquad (I_N)$$

for Eq. (E_N) to be oscillatory.

In 2002, Zhang and Deng [18] proved the following result for any time scale T.

THEOREM A (Corollary 2 in [18]). *Define*

$$\alpha := \limsup_{t\to\infty}\sup_{\lambda\in E}\{\lambda \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s\}$$

where $E = \{\ \lambda \mid \lambda > 0,\ 1 - \lambda p(t)\mu(t) > 0,\ t \in T\}$ *and*

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

If $\alpha < 1$, *then Eq.* (E) *is oscillatory.*

Later, in 2004, Bohner [3] proved Theorem A in a different way. In the same year, Zhang and Lian [19] studied the distribution of generalized zeros of solutions of the delay dynamic Eq. (E). Note that in [18] and [3], the conditions (I_R) and (I_N) are derived as a special case when $T = R$ and $T = N$, respectively.

It is obvious that there is a gap between the conditions (S_R) and (I_R) (or similarly between the conditions (S_N) and (I_N)) when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \qquad \left(or \quad \lim_{n\to\infty}\sum_{i=n-k}^{n-1} p_i\right)$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors. See [4,15-17] and the references cited therein.

The purpose of this paper is to establish new sufficient conditions for the oscillation of all solutions to dynamic equation (E). Moreover, the above mentioned problem is handled for some special cases of time scale T, and some results previously obtained are compared with the results presented in this paper. Several illustrative examples are given.

2 Main Results

Since we deal with the oscillatory behavior of the dynamic equation (E) on time scales, we assume throughout the paper that the time scale T under consideration satisfies $\sup T = \infty$. The following lemma is needed in our proofs.

LEMMA 1 *Assume that $f : T \to R$ is rd-continuous, $g : T \to R$ is nonincreasing and $\tau : T \to T$ is nondecreasing. If $v < u$, then*

$$\int_{v}^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_{v}^{\sigma(u)} f(s)\Delta s. \tag{5}$$

Proof: Since $v < u$, we can divide the integral into two parts

$$\int_{v}^{\sigma(u)} f(s)g(\tau(s))\Delta s = \int_{v}^{u} f(s)g(\tau(s))\Delta s + \int_{u}^{\sigma(u)} f(s)g(\tau(s))\Delta s.$$

Using the fact that τ is nondecreasing and g is nonincreasing, the first part gives

$$\int_{v}^{u} f(s)g(\tau(s))\Delta s \geq g(\tau(u)) \int_{v}^{u} f(s)\Delta s. \tag{6}$$

Since $fg(\tau)$ is rd-continuous, in view of (4), the second part yields

$$\int_u^{\sigma(u)} f(s)g(\tau(s))\Delta s = \mu(u)f(u)g(\tau(u)) = g(\tau(u))\Big(\mu(u)f(u)\Big) = g(\tau(u))\int_u^{\sigma(u)} f(s)\Delta s. \tag{7}$$

Combining (6) and (7), we obtain

$$\int_v^{\sigma(u)} f(s)g(\tau(s))\Delta s \geq g(\tau(u))\left(\int_v^{u} f(s)\Delta s + \int_u^{\sigma(u)} f(s)\Delta s\right) = g(\tau(u))\int_v^{\sigma(u)} f(s)\Delta s.$$

The proof is complete.

In case $v = u$, the monotonocity property of τ is not needed. In view of (7), the following corollary is immediate.

COROLLARY 1 *Assume that $f, g : T \to R$ are rd-continuous, g is nonincreasing and $\tau : T \to T$ is rd-continuous. Then*

$$\int_u^{\sigma(u)} f(s)g(\tau(s))\Delta s = g(\tau(u))\int_u^{\sigma(u)} f(s)\Delta s. \tag{8}$$

LEMMA 2 *(Cf. [3, 19]) Assume that $x(t)$ is an eventually positive solution of Eq. (E) and that for some positive constant M*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > M. \tag{9}$$

Then

$$\frac{x(\tau(t))}{x(t)} \leq \frac{4}{M^2}, \quad \textit{for all large } t. \tag{10}$$

Proof: For some sufficiently large t_0, $x(t) > 0$ when $t \geq t_0$. From Eq. (E), we have $x^{\Delta}(t) \leq 0$ for $t \geq \tau^{-1}(t_0) = t_1$. By (9), it is possible to find a sufficiently large number $t_2 \geq \tau^{-1}(t_1)$ such that

$$\int_{\tau(t)}^{t} p(s)\Delta s \geq M, \qquad \forall t \geq t_2. \tag{11}$$

Since $\tau^{-1}(t) > t \geq t_2$, by (11), we get

$$\int_{t}^{\tau^{-1}(t)} p(s)\Delta s \geq M, \quad \forall t \geq t_2. \tag{12}$$

Define

$$G(r) := \int_t^r p(s)\Delta s - \frac{M}{2},$$

for $r \in [t, \tau^{-1}(t)]$. It is clear that $G : [t, \tau^{-1}(t)] \to R$ is continuous and nondecreasing. We also have

$$G(t) = -\frac{M}{2} < 0 \qquad \text{and} \qquad G(\tau^{-1}(t)) \geq M - \frac{M}{2} = \frac{M}{2} > 0.$$

By Intermediate Value Theorem for time scales, there exists a $t_* \in [t, \tau^{-1}(t))$ such that $G(t_*)G(\sigma(t_*)) \leq 0$. Since G is nondecreasing, we conclude that $G(t_*) \leq 0 < G(\sigma(t_*))$. Hence there exists a $t_* \in [t, \tau^{-1}(t))$ such that

$$\int_t^{t_*} p(s)\Delta s \leq \frac{M}{2} \qquad \text{and} \qquad \int_t^{\sigma(t_*)} p(s)\Delta s > \frac{M}{2}, \quad \text{for} \quad t \geq t_2. \tag{13}$$

By (9) and the first part of (13), we also have

$$\int_{\tau(t_*)}^{\sigma(t)} p(s)\Delta s \geq \int_{\tau(t_*)}^{t_*} p(s)\Delta s - \int_t^{t_*} p(s)\Delta s \geq \frac{M}{2}, \quad \text{for} \quad t \geq t_2. \tag{14}$$

Using (5) and the decreasing property of x, we obtain

$$\int_{\tau(t_*)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \geq x(\tau(t))\frac{M}{2}, \tag{15}$$

and

$$\int_t^{\sigma(t_*)} p(s)x(\tau(s))\Delta s \geq x(\tau(t_*))\frac{M}{2}. \tag{16}$$

Integrating Eq. (E) from t to $\sigma(t_*)$ and using (15) and (16), we obtain for $t \geq t_2$,

$$\begin{aligned} x(t) &\geq x(t) - x(\sigma(t_*)) = \int_t^{\sigma(t_*)} p(s)x(\tau(s))\Delta s \\ &\geq \frac{M}{2}x(\tau(t_*)) \geq \frac{M}{2}[x(\tau(t_*)) - x(\sigma(t))] = \frac{M}{2}\int_{\tau(t_*)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \\ &\geq \frac{M^2}{4}x(\tau(t)). \end{aligned}$$

The proof is complete.

Note that Lemma 2 is a generalization of results in [9] and [6].

THEOREM 1 *If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{17}$$

then Eq. (E) is oscillatory.

Proof: Assume, for the sake of contradiction, that Eq. (E) has a nonoscillatory solution $x(t)$. We may assume that $x(t)$ is eventually positive by replacing x by $-x$, otherwise. Since $\tau(t)\to\infty$ as $t\to\infty$, there is a positive number $t_1 \geq t_0$, such that $x(\tau(t)) > 0$ for $t \geq t_1$. In view of Eq. (E),

$$x^{\Delta}(t) = -p(t)x(\tau(t)) < 0, \qquad t \geq t_1. \tag{18}$$

Integrating Eq. (E) from $\tau(t)$ to $\sigma(t)$, we have

$$x(\sigma(t)) - x(\tau(t)) + \int_{\tau(t)}^{\sigma(t)} p(s)x(\tau(s))\Delta s = 0. \tag{19}$$

Since x is Δ-differentiable, it is rd-continuous, Lemma 1 is applicable for the integral term in the previous equation. In view of (5), it is easy to see that

$$\int_{\tau(t)}^{\sigma(t)} p(s)x(\tau(s))\Delta s \geq x(\tau(t)) \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s. \tag{20}$$

Using (20) in (19), we obtain

$$x(\sigma(t)) + x(\tau(t))\Big(\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s - 1\Big) \leq 0, \qquad \text{for} \qquad t \geq t_1, \tag{21}$$

which, in view of the condition (17), leads to a contradiction. The proof is complete.

Observe that Theorem 1 unifies previous results related with the oscillation of first order delay equations in the continuous and discrete case. In particular, if $T = R$ or $T = N$, condition (17) of Theorem 1 takes the form (S_R) or (S_N), respectively.

THEOREM 2 *Assume that there exists a positive constant M such that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > M \tag{22}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{M^2}{4}. \tag{23}$$

Then Eq. (E) is oscillatory.

Proof: Assume that $x(t)$ is an eventually positive solution of Eq. (E) such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. As in the proof of Theorem 1, integrating Eq.

(E) from $\tau(t)$ to t, we have

$$\begin{aligned} 0 &= x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)x(\tau(s))\Delta s \\ &\geq x(t) + x(\tau(t))\Big(\int_{\tau(t)}^{t} p(s)\Delta s - 1\Big) \\ &= x(\tau(t))\Big(\frac{x(t)}{x(\tau(t))} + \int_{\tau(t)}^{t} p(s)\Delta s - 1\Big), \qquad \text{for} \qquad t \geq t_1. \end{aligned}$$

Now, using Lemma 2, we obtain

$$x(\tau(t))\Big(\frac{M^2}{4} + \int_{\tau(t)}^{t} p(s)\Delta s - 1\Big) \leq 0, \qquad \text{for all} \qquad t \geq t_1,$$

which, in view of condition (23), leads to a contradiction. The proof is complete. Note that, in view of Theroem 1, it makes sense to consider in Theorem 2 the case when $0 < M < 1$. Also observe that Theorem 2.2 in [4] for Eq. (E_R) can be derived from Theorem 2 when the time scale T is chosen as R.

Consider the time scale of the form

$$T = \{t_n : n \in Z\}, \tag{24}$$

where $\{t_n\}$ is a strictly increasing sequence of real numbers such that T is closed. For such time scales, Bohner [3] presented the following result.

Theorem B (Theorem 2, [3]). *Consider a time scale as described in* (24). *Let* $k \in N$ *and* $\tau(t_n) = t_{n-k}$ *for all* $n \in Z$. *If Eq. (E) has an eventually positive solution, then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \leq (\frac{k}{k+1})^{k+1}.$$

An immediate consequence is the following result.

Corollary 2 *Consider a time scale as described in (24). Let* $k \in N$ *and* $\tau(t_n) = t_{n-k}$ *for all* $n \in Z$. *If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > (\frac{k}{k+1})^{k+1}, \tag{25}$$

then Eq. (E) is oscillatory.

Observe that when condition (25) is not satisfied, then from Corollary 2, we can not conclude anything about the oscillatory behavior of Eq. (E). However, from Theorem 2, we have the following conclusion.

COROLLARY 3 *Assume that there exists a positive real number M such that*

$$M < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \le (\frac{k}{k+1})^{k+1} \tag{26}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{M^2}{4}.$$

Then Eq. (E) is oscillatory on the time scales described in (24) with $k \in N$ and $\tau(t_n) = t_{n-k}$ for all $n \in N$.

Note that for Eq. (E_N) Theorem 2.6 in [17] can be derived from Corollary 3 when the time scale T is chosen as N.

3 Examples

EXAMPLE 1 *Let $T = \{t \mid \mu(t) \equiv \mu$, a positive constant $\forall t\}$. Consider the following equation*

$$x^{\Delta}(t) + p\, x(\tau(t)) = 0, \qquad t \ge t_0, \tag{27}$$

where $p > 0$ and $\tau(t) = t - (k-1)\mu$, for any integer $k > 1$. Since

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p\ \Delta s = p\ k\mu,$$

we conclude that if

$$pk\mu > 1,$$

then, by Theorem 1, all solutions of Eq.(27) are oscillatory.

EXAMPLE 2 *Assume that in equation (27), $p = 1$, $k = 9$, and $\mu = \frac{1}{10}$. Since $pk\mu = 9/10 \not> 1$, Theorem 1 can not be applied. However, it is easy to see that there exists a positive real number $M \in (\sqrt{2/5}, 9/10)$ so that (22) and the condition (23)*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s = pk\mu = \frac{9}{10} > 1 - \frac{M^2}{4}$$

are satisfied. So, by Theorem 2, all solutions of Eq. (27) are oscillatory.

EXAMPLE 3 *(See [17]). Consider the equation on $T = N$*

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \qquad n = 0, 1, 2, ..., \tag{28}$$

where

$$p_{2n} = \frac{8}{100}, \quad p_{2n+1} = \frac{8}{100} + \frac{746}{1000} sin^2 \frac{n\pi}{2}, \qquad n = 0, 1, 2, ...$$

Then

$$\liminf_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} < (\frac{3}{4})^4,$$

which means that the condition (25) of Corollary 2 is not satisfied. However, it is easy to see that there exists a positive real number $M \in (\sqrt{7/5^3}, 24/100)$ such that

$$\limsup_{n \to \infty} \sum_{i=n-3}^{n-1} p_i = \frac{24}{100} + \frac{746}{1000} > 1 - \frac{M^2}{4}.$$

So, by Corollary 3, all solutions of Eq. (28) are oscillatory.

EXAMPLE 4 *Consider the delay difference equation*

$$x_{n+1} - x_n + p_n x_{n-1} = 0, \tag{29}$$

where $$p_{2n} = \frac{1}{5}, \qquad p_{2n+1} = \frac{127}{128}, \qquad n = 0, 1, 2, ...,$$

By Theorem A,

$$\begin{aligned} \alpha &= \limsup_{t \to \infty} \sup_{\lambda \in E} \lambda \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s)) \Delta s \\ &= \limsup_{t \to \infty} \sup_{\lambda \in E} \lambda \exp \int_{\tau(t)}^{t} \frac{Log(1 - \lambda p(s)\mu(s))}{\mu(s)} \Delta s \\ &= \limsup_{t \to \infty} \sup_{\lambda \in E} \lambda \exp \sum_{i=n-1}^{n-1} Log(1 - \lambda p_i) \\ &= \limsup_{t \to \infty} \sup_{\lambda \in E} \lambda(1 - \lambda p_{n-1}) \\ &= \limsup_{t \to \infty} \frac{1}{4p_{n-1}}, \end{aligned}$$

because $\lambda(1 - \lambda p_{n-1})$ takes its maximum value at $\lambda = \dfrac{1}{2p_{n-1}}$. Thus

$$\alpha = \limsup_{t \to \infty} \frac{1}{4p_{n-1}} = \frac{5}{4} > 1,$$

and therefore, Theorem A can not be applied.

Also

$$\liminf_{t\to\infty} \sum_{i=n-1}^{n-1} p_i = \frac{1}{5} < \left(\frac{1}{2}\right)^2,$$

that is, condition (25) is not satisfied and therefore Corollary 2 can not be applied.

However

$$\limsup_{t\to\infty} \sum_{i=n-1}^{n-1} p_i = \frac{127}{128},$$

and taking $M \in (1/4\sqrt{2}, 1/5)$, *conditions (22) and (23) of Theorem 2 are satisfied. Therefore, all solutions of Eq. (29) are oscillatory.*

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ON SOME SECOND ORDER NONLOCAL FUNCTIONAL AND ORDINARY BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper we prove the existence of multiple positive solutions for second order nonlinear functional and ordinary nonlocal boundary value problems. The results are obtained by using a fixed point theorem on a Banach space, ordered by an appropriate cone.

1. INTRODUCTION

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ =: [0, +\infty)$ and $I =: [0, 1]$. Also, let $q \in [0, 1)$ and $J := [-q, 0]$. For every closed interval $B \subseteq J \cup I$ we denote by $C(B)$ the Banach space of all continuous real functions $\psi : B \to \mathbb{R}$ endowed with the usual sup-norm

$$\|\psi\|_B := \sup\{|\psi(s)| : s \in B\}.$$

Also, we define the set $C^+(B)$ as follows

$$C^+(B) := \{\psi \in C(B) : \psi \geq 0\}.$$

If $x \in C(J \cup I)$ and $t \in I$, then we denote by x_t the element of $C(J)$ defined by

$$x_t(s) = x(t + s), \quad s \in J.$$

Now, consider the equation

$$x''(t) + f(t, x_t) = 0, \quad t \in I, \tag{1.1}$$

along with the boundary conditions

$$x_0 = \phi, \tag{1.2}$$

and

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{1.3}$$

where $f : \mathbb{R}^+ \times C^+(J) \to \mathbb{R}^+$ and $\phi : J \to \mathbb{R}^+$ are continuous functions, $g : I \to \mathbb{R}$ is a nondecreasing function, such that $g(0) = 0$ and $1 - g(1) > 0$.

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The problem of existence of positive solutions for boundary value problems for second order differential equations which involve a nonlocal condition like (1.3) has been treated recently by Karakostas and Tsamatos [8-11] and Tsamatos [20]. Moreover, boundary value problems with integral boundary conditions for second order differential equations with retarded arguments is the subject of the papers [12] and [17]. In the recent years an increasing interest is also observed for boundary value problems concerning functional differential equations (see [5,6] and the references therein). Fixed point theorems on Banach spaces ordered by appropriate cones is usually the tool for proving multiple positive solutions for boundary value problems. The famous Guo - Krasnoselskii fixed point theorem [13,22] seems to be used in the largest part of the papers on this subject. Also the well known Leggett-Williams fixed point theorem [14] and some recent generalizations of it are used in proving multiple positive solutions for various types of boundary value problems.

For a detailed exposition of the theory of functional differential equations, like (1.1), the reader can refer to the books due to Hale and Lunel [4] and Azbelev *et all.* [3].

In this paper, we choose to use a fixed point theorem, on an ordered by cones Banach space, due to Avery and Henderson [1] (see also [15,19])which, apart from guarantying the existence of two positive solutions, has the advantage to offer some additional information on these solutions. In our results, the values of these solutions at three given points of their domain are upper or lower bounded by a-priori given constants. We note that this fixed point theorem was used recently in several papers (see [2,5,15,16,18,19] and the references therein).

Since the results we present are new even in the ordinary case, we mention them for this case too, underlining the necessary adjustments that have to be made to the hypothesis referring to the functional case.

The paper is organized as follows. In section 2 we present the definitions and the lemmas we are going to use, as well as the fixed point theorem, on which we base our results. In section 3, we present the new results for the functional case and then in section 4 the results for the ordinary case. Finally, in section 5 we give some applications of our results.

2. Preliminaries and some basic lemmas

Definition. *A function $x \in C(J \cup I)$ is a solution of the boundary value problem* $(1.1)-(1.3)$ *if x satisfies equation* (1.1)*, the boundary condition* (1.3) *and, moreover* $x|J = \phi$.

Lemma 2.1. *A function $x \in C(J \cup I)$ is a solution of the boundary value problem* $(1.1)-(1.3)$ *if and only if x is a fixed point of the operator $A : C(J \cup I) \to C(J \cup I)$, with*

$$Ax(t) = \begin{cases} \phi(t), & t \in J \\ \phi(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r) dr dg(s) + \int_0^t \int_s^1 f(r, x_r) dr ds, & t \in I, \end{cases}$$

where $\zeta := \frac{1}{1-g(1)}$.

Proof. Suppose that x is a solution of the boundary value problem $(1.1)-(1.3)$. Then, obviously, $x|J = \phi$. Moreover, by integrating (1.1) we get

$$x'(t) = x'(1) + \int_t^1 f(s, x_s) ds, \quad t \in I. \tag{2.1}$$

Also from (1.3) and (2.1) we have

$$\begin{aligned} x'(1) &= \int_0^1 x'(s)dg(s) \\ &= \int_0^1 \left(x'(1) + \int_s^1 f(r, x_r)dr \right) dg(s) \\ &= x'(1) \int_0^1 dg(s) + \int_0^1 \int_s^1 f(r, x_r)drdg(s). \end{aligned}$$

Therefore

$$x'(1) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s). \tag{2.2}$$

Combining (2.1) and (2.2) we conclude that

$$x'(t) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_t^1 f(s, x_s)ds, \ \ t \in I,$$

which by integration from 0 to t, $t \in I$ gives

$$\begin{aligned} x(t) &= x(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^t \int_s^1 f(r, x_r)drds \\ &= \phi(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^t \int_s^1 f(r, x_r)drds. \end{aligned}$$

The above step gives that, if x is a solution of the boundary value problem (1.1) – (1.3), then $x = Ax$.

For the inverse, suppose that x is a fixed point of the operator A. Then, obviously, $\phi(t) = x(t) = x(0+t) = x_0(t)$ for $t \in J$. Also from the form of A we have

$$x'(t) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_t^1 f(r, x_r)dr, \ \ t \in I. \tag{2.3}$$

Therefore

$$x'(1) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s). \tag{2.4}$$

Then from (2.3), (2.4) and the fact that $\zeta := \frac{1}{1-g(1)}$, $g(0) = 0$, we get

$$x'(t) = x'(1) + \int_t^1 f(s, x_s)ds, \ \ t \in I. \tag{2.5}$$

Using (2.4) and (2.5) we have

$$\begin{aligned} x'(1) &= x'(1)g(1) + \int_0^1 \int_s^1 f(r, x_r)drdg(s) \\ &= x'(1) \int_0^1 dg(s) + \int_0^1 \int_s^1 f(r, x_r)drdg(s) \\ &= \int_0^1 \left(x'(1) + \int_s^1 f(r, x_r)dr \right) dg(s) \\ &= \int_0^1 x'(s)dg(s). \end{aligned}$$

Finally, from (2.5) we have

$$x'(t) = x'(1) - \int_1^t f(s, x_s)ds, \;\; t \in I$$

and so

$$x''(t) + f(t, x_t) = 0, \;\; t \in I.$$

The proof is complete. $\square$

The following lemma can be found in [21].

Lemma 2.2. *Let function $x \in C(I)$ be concave and non negative and $\xi \in (0, \frac{1}{2})$. Then*

$$(1) \;\; x(t) \geq \begin{cases} \frac{\|x\|t}{\sigma}, & 0 \leq t \leq \sigma, \\ \|x\| \frac{1-t}{1-\sigma}, & \sigma \leq t \leq 1, \end{cases} \quad \textit{if } 0 < \sigma < 1,$$

(2) $x(t) \geq \|x\|t,\ 0 \leq t \leq 1,$ *if* $\sigma = 1,$

(3) $x(t) \geq \|x\|(1-t),\ 0 \leq t \leq 1,$ *if* $\sigma = 0,$

(4) $x(t) \geq \xi\|x\|,$ *for all* $t \in [\xi, 1-\xi],$

where $\|x\| := \sup\{|x(t)| : 0 \leq t \leq 1\}$ *and* $\sigma \in [0,1]$ *such that* $x(\sigma) = \|x\|.$

The results proved in this paper are based on the following Theorem 2.6 due to R. I. Avery and J. Henderson [1] (see also [15] and [19]). As we mentioned in the introduction, this theorem ensures that our boundary value problem (1.1) – (1.3) has at least two distinct positive solutions and, moreover, for each of these solutions, we have an upper bound at some specific point of its domain and a lower bound at some other specific point of its domain. Also, both solutions are concave and nondecreasing on I. In order to apply this theorem some definitions are necessary.

Definition 2.3. *Let $\mathbb{E}$ be a real Banach space. A cone in $\mathbb{E}$ is a nonempty, closed set $\mathbb{P} \subset \mathbb{E}$ such that*

(*i*) $\kappa u + \lambda v \in \mathbb{P}$ *for all* $u,\ v \in \mathbb{P}$ *and all* $\kappa,\ \lambda \geq 0$

(*ii*) $u,\ -u \in \mathbb{P}$ *implies* $u = 0.$

Definition 2.4. *Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{B}$. A functional $\psi : \mathbb{P} \to \mathbb{B}$ is said to be increasing on $\mathbb{P}$ if $\psi(x) \leq \psi(y)$, for any $x,\ y \in \mathbb{P}$ with $x \leq y$, where $\leq$ is the partial ordering induced to the Banach space by the cone $\mathbb{P}$, i.e.*

$$x \leq y \;\; \textit{if and only if} \;\; y - x \in \mathbb{P}.$$

Definition 2.5. *Let ψ be a nonnegative functional on a cone $\mathbb{P}$. For each $d > 0$ we denote by $\mathbb{P}(\psi, d)$ the set*

$$\mathbb{P}(\psi, d) := \{x \in \mathbb{P} : \psi(x) < d\}.$$

Theorem 2.6. *Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{E}$. Let α and γ be increasing, nonnegative, continuous functionals on $\mathbb{P}$, and let θ be a nonnegative functional on $\mathbb{P}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $\Theta > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq \Theta\gamma(x),$$

for all $x \in \overline{\mathbb{P}(\gamma, c)}$. Suppose there exists a completely continuous operator $A : \overline{\mathbb{P}(\gamma, c)} \to \mathbb{P}$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \text{for} \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad x \in \vartheta\mathbb{P}(\theta, b),$$

and

(*i*) $\gamma(Ax) > c$, *for all $x \in \partial\mathbb{P}(\gamma, c)$,*

(*ii*) $\theta(Ax) < b$, *for all $x \in \partial\mathbb{P}(\theta, b)$,*

(*iii*) $\mathbb{P}(\alpha, a) \neq \emptyset$, *and $\alpha(Ax) > a$, for all $x \in \partial\mathbb{P}(\alpha, a)$,*

or

(*i'*) $\gamma(Ax) < c$, *for all $x \in \partial\mathbb{P}(\gamma, c)$,*

(*ii'*) $\theta(Ax) > b$, *for all $x \in \partial\mathbb{P}(\theta, b)$,*

(*iii'*) $\mathbb{P}(\alpha, a) \neq \emptyset$, *and $\alpha(Ax) < a$, for all $x \in \partial\mathbb{P}(\alpha, a)$.*

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{\mathbb{P}(\gamma, c)}$ such that

$$a < \alpha(x_1), \quad \text{with} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \text{with} \quad \gamma(x_2) < c.$$

3. Main Results

Define the set

$$\mathbb{K} := \{x \in C(J \cup I) : x(t) \geq 0, \ t \in J \cup I, \ x/I \text{ is concave and nondecreasing}\},$$

which is a cone in $C(J \cup I)$. Also let

$$0 < r_1 \leq r_2 \leq r_3 \leq 1$$

and consider the following functionals

$$\gamma(x) = x(r_1), \ x \in \mathbb{K},$$

$$\theta(x) = x(r_2), \ x \in \mathbb{K}$$

and

$$\alpha(x) = x(r_3), \ x \in \mathbb{K}.$$

It is easy to see that α, γ are nonnegative, increasing and continuous functionals on $\mathbb{K}$, θ is nonnegative on $\mathbb{K}$ and $\theta(0) = 0$. Also, it is straightforward that

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \tag{3.1}$$

since $x \in \mathbb{K}$ is nondecreasing on I. Furthermore, for any $x \in \mathbb{K}$, by Lemma 2.2 (inequality (2)), we have

$$\gamma(x) = x(r_1) \geq r_1 \|x\|_I.$$

So

$$\|x\|_I \leq \frac{1}{r_1}\gamma(x), \quad x \in \mathbb{K}. \tag{3.2}$$

Additionally, by the definition of θ it is obvious that

$$\theta(\lambda x) = \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in \mathbb{K}.$$

Now, if $D \subset I$, consider the functions $H : C(I) \to C(I)$ and $H_D : C(I) \to C(I)$ by

$$(Hz)(s) := \int_s^1 z(r)dr, \quad s \in I$$

and

$$(H_D z)(s) := \int_{D\cap[s,1]} z(r)dr, \quad s \in I.$$

At this point, we state the following assumptions:

(H_1) There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L(\|y\|_J), \quad t \in I, \quad y \in C^+(J)$$

and also

$$\phi(0) + L(M)\left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2} (Hu)(s)ds\right) < Mr_2.$$

(H_2) There exist a constant $\delta \in (0, 1)$ and functions $\tau : I \to [0, q]$, continuous $v : I \to \mathbb{R}^+$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y(-\tau(t))), \quad t \in X, \quad y \in \{h \in C^+(J) : \|h\|_J < M\},$$

where

$$X := \{t \in I : \delta \leq t - \tau(t) \leq 1\},$$

$\sup\{v(t) : t \in X\} > 0$, $meas(X \cap [s, 1)) > 0$ for all $s \in [0, 1)$ and M is defined in (H_1).

(H_3) There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta} < \phi(0) + w(\rho_i)\left(r_i\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_i} (H_X v)(s)ds\right), \quad i = 1, 3,$$

and

$$\frac{\rho_3}{\delta} < Mr_2 < \frac{\rho_1}{\delta}.$$

Notice that if $\phi(0) \neq 0$, then these ρ_1, ρ_3 always exist.

Remark. *It easy to see that* $\sup\{v(t) : t \in X\} > 0$ *and* $meas(X \cap [s, 1)) > 0,\ s \in [0, 1)$ *in the assumption* (H_2), *imply that* $\int_0^1 (H_X v)(s) dg(s) > 0$ *and* $\int_0^{r_i} (H_X v)(s) ds > 0,\ i = 1, 3$.

Theorem 3.1. *Suppose that assumptions* $(H_1)-(H_3)$ *hold and furthermore* $\|\phi\|_J \le M$. *Then the boundary value problem* $(1.1) - (1.3)$ *has at least two concave and nondecreasing on* I *and positive on* $J \cup I$ *solutions* x_1, x_2 *such that* $x_1(r_3) > \frac{\rho_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ *and* $x_2(r_1) < \frac{\rho_1}{\delta}$.

Proof. First of all, we observe that, because of (H_1), $f(t, \cdot)$ maps bounded sets into bounded sets. Therefore A is a completely continuous operator.

Now we set $a = \frac{\rho_3}{\delta}$, $b = Mr_2$, $c = \frac{\rho_1}{\delta}$ and we consider a $x \in \overline{\mathbb{K}(\gamma, c)}$. Then since $\zeta > 0$ and $f(t, x_t) \ge 0$ for every $t \in I$, we get that $Ax(t) \ge 0,\ t \in I$. Also $Ax(t) = \phi(t) \ge 0,\ t \in J$. Thus $Ax(t) \ge 0,\ t \in J \cup I$. Moreover, $(Ax)''(t) = -f(t, x_t) \le 0$, which means that Ax is concave on I. Also it is clear that $(Ax)'(t) \ge 0$ for $t \in I$. So $A : \overline{\mathbb{K}(\gamma, c)} \to \mathbb{K}$.

Now let $x \in \partial\mathbb{K}(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$ and so $\|x\|_I \ge c$. Having in mind assumption (H_2), we get

$$\begin{aligned}
\gamma(Ax) = Ax(r_1) &= \phi(0) + \zeta r_1 \int_0^1 \int_s^1 f(r, x_r) dr dg(s) + \int_0^{r_1} \int_s^1 f(r, x_r) dr ds \\
&\ge \phi(0) + \zeta r_1 \int_0^1 \int_{X \cap [s,1]} f(r, x_r) dr dg(s) + \int_0^{r_1} \int_{X \cap [s,1]} f(r, x_r) dr ds \\
&\ge \phi(0) + \zeta r_1 \int_0^1 \int_{X \cap [s,1]} v(r) w(x_r(-\tau(r))) dr dg(s) \\
&\quad + \int_0^{r_1} \int_{X \cap [s,1]} v(r) w(x_r(-\tau(r))) dr ds \\
&= \phi(0) + \zeta r_1 \int_0^1 \int_{X \cap [s,1]} v(r) w(x(r - \tau(r))) dr dg(s) \\
&\quad + \int_0^{r_1} \int_{X \cap [s,1]} v(r) w(x(r - \tau(r))) dr ds \\
&\ge \phi(0) + \zeta r_1 \int_0^1 \int_{X \cap [s,1]} v(r) w(x(\delta)) dr dg(s) \\
&\quad + \int_0^{r_1} \int_{X \cap [s,1]} v(r) w(x(\delta)) dr ds.
\end{aligned}$$

Additionally, by assumption (H_3) and inequality 2 of Lemma 2.2, we have

$$\begin{aligned}
\gamma(Ax) &\ge \phi(0) + w(\delta \|x\|_I) \left(r_1 \zeta \int_0^1 (H_X v)(s) dg(s) + \int_0^{r_1} (H_X v)(s) ds \right) \\
&\ge \phi(0) + w(\delta c) \left(r_1 \zeta \int_0^1 (H_X v)(s) dg(s) + \int_0^{r_1} (H_X v)(s) ds \right) \\
&= \phi(0) + w(\rho_1) \left(r_1 \zeta \int_0^1 (H_X v)(s) dg(s) + \int_0^{r_1} (H_X v)(s) ds \right) \\
&> \frac{\rho_1}{\delta} = c.
\end{aligned}$$

This means that condition (i) of Theorem 2.5 is satisfied.

Now let $x \in \partial\mathbb{K}(\theta, b)$. Then $\theta(x) = x(r_2) = b$ and so by inequality 2 of Lemma 2.2 we get

$$\|x\|_I \leq \frac{1}{r_2}x(r_2) = \frac{1}{r_2}\theta(x) = \frac{b}{r_2}.$$

Also we assumed that $\|\phi\|_J \leq M = \frac{b}{r_2}$, so $\|x\|_{J\cup I} \leq \frac{b}{r_2}$. Now, by (H_1), we have

$$\begin{aligned}
\theta(Ax) &= Ax(r_2) \\
&= \phi(0) + \zeta r_2 \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^{r_2} \int_s^1 f(r, x_r)drds \\
&\leq \phi(0) + \zeta r_2 \int_0^1 \int_s^1 u(r)L(\|x_r\|_J)drdg(s) + \int_0^{r_2} \int_s^1 u(r)L(\|x_r\|_J)drds \\
&\leq \phi(0) + \zeta r_2 \int_0^1 \int_s^1 u(r)L(\frac{b}{r_2})drdg(s) + \int_0^{r_2} \int_s^1 u(r)L(\frac{b}{r_2})drds \\
&= \phi(0) + L(M)\left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2} (Hu)(s)ds\right) \\
&< Mr_2 = b.
\end{aligned}$$

So condition (ii) of Theorem 2.5 is also satisfied.

Now, define the function $y : J \cup I \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $\mathbb{K}(\alpha, a) \neq \emptyset$. Also, for any $x \in \partial\mathbb{K}(\alpha, a)$ we have $\alpha(x) = x(r_3) = a$. Therefore $\|x\|_I \geq a$. Now, having in mind assumption (H_2) and as in the case of the functional γ above, we get

$$\begin{aligned}
\alpha(Ax) &= Ax(r_3) \\
&\geq \phi(0) + \zeta r_3 \int_0^1 \int_{X\cap[s,1]} v(r)w(x(\delta))drdg(s) \\
&\quad + \int_0^{r_3} \int_{X\cap[s,1]} v(r)w(x(\delta))drds.
\end{aligned}$$

Then, by assumption (H_3) and inequality 2 of Lemma 2.2, we also have

$$\alpha(Ax) \geq \phi(0) + w(\delta a)\left(r_3\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_3} (H_X v)(s)ds\right) > \frac{\rho_3}{\delta} = a.$$

Consequently, assumption (iii) of Theorem 2.5 is satisfied.

The result can now be obtained by applying Theorem 2.5. □

The above Theorem 3.1 has been obtained by using the requirements $(i) - (iii)$ of Theorem 2.5. Using the requirements $(i') - (iii')$ of the same theorem we can also obtain another existence theorem (Theorem 3.2 below) for our boundary value problem (1.1) – (1.3). For this purpose we need the following assumptions.

- $(\widehat{H}_1)$ There exist $M_1, M_3 > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L(\|y\|_J), \quad t \in I, \quad y \in C^+(J)$$

and also

$$\phi(0) + L(M_i)\left(\zeta r_i \int_0^1 (Hu)(s)dg(s) + \int_0^{r_i} (Hu)(s)ds\right) < M_i r_i, \ \ i = 1, 3.$$

($\widehat{H}_2$) There exist a constant $\delta \in (0,1)$ and functions $\tau : I \to [0, q]$, continuous $v : I \to \mathbb{R}^+$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y(-\tau(t))), \ \ t \in X, \ \ y \in \{h \in C^+(J) : \|h\|_J < \min\{M_1, M_3\}\},$$

where

$$X := \{t \in I : \delta \leq t - \tau(t) \leq 1\},$$

$\sup\{v(t) : t \in X\} > 0$, $meas(X \cap [s, 1]) > 0$ for all $s \in [0, 1)$ and M is defined in (H_4).

($\widehat{H}_3$) There exists $\rho > 0$ such that

$$\frac{\rho}{\delta} < \phi(0) + w(\rho)\left(r_2\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_2} (H_X v)(s)ds\right).$$

Notice that if $\phi(0) \neq 0$, then this ρ always exists.

Theorem 3.2. *Suppose that assumptions ($\widehat{H}_1$)–($\widehat{H}_3$) hold and furthermore $\|\phi\|_J \leq \min\{M_1, M_3\}$.*

Then the boundary value problem (1.1) – (1.3) *has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2 such that $x_1(r_3) > M_3 r_3$, $x_1(r_2) < \frac{\rho}{\delta}$, $x_2(r_2) > \frac{\rho}{\delta}$ and $x_2(r_1) < M_1 r_1$.*

Proof. Consider the functionals γ, θ, α as in Theorem 3.1. Our purpose is to prove that requirements $(i'), (ii'), (iii')$ are satisfied.

Let $a = M_3 r_3$, $b = \frac{\rho}{\delta}$, $c = M_1 r_1$ and $x \in \partial\mathbb{K}(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$ and, by Lemma 2.2 (inequality 2), we get

$$\|x\|_I \leq \frac{1}{r_1}x(r_1) = \frac{1}{r_1}\gamma(x) = \frac{c}{r_1}.$$

Also we assumed that $\|\phi\|_J \leq M_1 = \frac{c}{r_1}$, so $\|x\|_{J \cup I} \leq \frac{c}{r_1}$. Then, by ($\widehat{H}_1$) and following the same arguments as in the proof of Theorem 3.1 we can easily prove that

$$\gamma(Ax) < c.$$

So condition (i') of Theorem 2.5 is satisfied.

Now let $x \in \partial\mathbb{K}(\theta, b)$. Then $\theta(x) = x(r_2) = b$, so $\|x\|_I \geq b$. Hence, having in mind assumptions ($\widehat{H}_2$), ($\widehat{H}_3$) and inequality 2 of Lemma 2.2, as in the proof of Theorem 3.1, we can prove that

$$\theta(Ax) > b.$$

This means that condition (ii') of Theorem 2.5 is satisfied.

Now, define the function $y : J \cup I \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $\mathbb{K}(\alpha, a) \neq \emptyset$. Also let $x \in \partial\mathbb{K}(\alpha, a)$. Then $\alpha(x) = x(r_3) = a$. So, by Lemma 2.2 (inequality 2) we get

$$\|x\|_I \leq \frac{1}{r_3}x(r_3) = \frac{1}{r_3}\alpha(x) = \frac{a}{r_3}.$$

Also we assumed that $\|\phi\|_J \leq M_3 = \frac{a}{r_3}$, so $\|x\|_{J\cup I} \leq \frac{a}{r_3}$. Now, by $(\widehat{H}_1)$, we can easily prove that

$$\alpha(Ax) < a.$$

Consequently, assumption (iii') of Theorem 2.5 is satisfied and our proof is completed. $\square$

The obtained solutions x_1, x_2 in Theorems 3.1, 3.2 above are all nondecreasing. Thus, in the special case when $r_1 = r_2 = r_3 = 1$, we have that $x_i(r_j) = x_i(1) = \|x_i\|$, $i = 1, 2$, $j = 1, 2, 3$. Therefore, we have the following corollary of Theorems 3.1 and 3.2.

Corollary 3.3. *Suppose that assumptions* $(H_1) - (H_3)$ *(resp.* $(\widehat{H}_1) - (\widehat{H}_3)$*) hold and furthermore* $\|\phi\|_J \leq M$ *(resp.* $\|\phi\|_J \leq \min\{M_1, M_3\}$*). Then the boundary value problem* (1.1) − (1.3) *has at least two concave and nondecreasing on* I *and positive on* $J \cup I$ *solutions* x_1, x_2 *such that*

$$\frac{\rho_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\rho_1}{\delta}$$

(resp. $M_3 < \|x_1\| < \frac{\rho}{\delta} < \|x_2\| < M_1$*).*

It is remarkable to observe that this corollary can be also obtained by applying twice the Krasnoselkii's theorem under the same assumptions $(H_1) - (H_3)$ (*resp.* $(\widehat{H}_1) - (\widehat{H}_3)$).

4. The Ordinary Case

In this section we suppose that $q = 0$. Then $J = \{0\}$, so the boundary value problem (1.1) − (1.3) is reformulated as follows

$$x''(t) + f(t, x(t)) = 0, \ \ t \in I, \tag{4.1}$$

$$x(0) = N, \tag{4.2}$$

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{4.3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $g : I \to \mathbb{R}^+$ is a nondecreasing function, such that $g(0) = 0$, $1 - g(1) > 0$ and $N \in \mathbb{R}^+$. Note that equation (4.1) is equivalent to the following form

$$x''(t) + f(t, x_t(0)) = 0, \ \ t \in I$$

and $C^+(\{0\}) \equiv \mathbb{R}^+$, so $f : \mathbb{R}^+ \times C^+(\{0\}) \to \mathbb{R}^+$.

Now, the analogue of Lemma 2.1 for this case is the following

Lemma 4.1. *A function $x \in C(I)$ is a solution of the boundary value problem (4.1) – (4.3) if and only if x is a fixed point of the operator $\widehat{A} : C(I) \to C(I)$, with*

$$\widehat{A}x(t) = N + \zeta t \int_0^1 \int_s^1 f(r, x(r))drdg(s) + \int_0^t \int_s^1 f(r, x(r))drds, \quad t \in I,$$

where $\zeta := \frac{1}{1-g(1)}$.

Assumptions $(H_1) - (H_3)$ and $(\widehat{H}_1) - (\widehat{H}_3)$, for the special case $q = 0$, are stated as follows:

$(H_1)_0$ There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L(y), \quad t \in I, y \in \mathbb{R}^+$$

and

$$N + L(M) \left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2} (Hu)(s)ds \right) \leq M r_2,$$

where the function H is defined in the previous section.

$(H_2)_0$ There exist $\delta \in (0, 1)$ and functions $v : I \to \mathbb{R}^+$ continuous and $w : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that

$$f(t, y) \geq v(t)w(y), \quad t \in Z := [\delta, 1], \quad y \in [0, M]$$

and $\sup\{v(t) : t \in Z\} > 0$.

$(H_3)_0$ There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta} \leq N + w(\rho_i) \left(r_i \zeta \int_0^1 (H_Z v)(s)dg(s) + \int_0^{r_i} (H_Z v)(s)ds \right), \quad i = 1, 3,$$

where the function H_Z is defined in previous section.

$(\widehat{H}_1)_0$ There exist $M_1, M_3 > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L(y), \quad t \in I, \quad y \in \mathbb{R}^+$$

and

$$N + L(M_i) \left(\zeta r_i \int_0^1 (Hu)(s)dg(s) + \int_0^{r_i} (Hu)(s)ds \right) < M_i r_i, \quad i = 1, 3$$

where the function H is defined in the previous section.

$(\widehat{H}_2)_0$ There exist a constant $\delta \in (0, 1)$, continuous function $v : I \to \mathbb{R}^+$ and nondecreasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y), \quad t \in Z := [\delta, 1], \quad y \in \mathbb{R}^+$$

and $\sup\{v(t) : t \in Z\} > 0$.

$(\widehat{H}_3)_0$ There exists $\rho > 0$ such that

$$\frac{\rho}{\delta} < N + w(\rho) \left(r_2 \zeta \int_0^1 (H_Z v)(s)dg(s) + \int_0^{r_2} (H_Z v)(s)ds \right),$$

where the function H_Z is defined in the previous section.

Therefore, we have the following theorems, which are the analogue of Theorems 3.1 and 3.2 respectively:

Theorem 4.2. *Suppose that assumptions* $(\widehat{H}_1)-(\widehat{H}_3)$ *hold and furthermore* $N \le M$. *Then the boundary value problem* (4.1) − (4.3) *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that* $x_1(r_3) > \frac{\rho_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ *and* $x_2(r_1) < \frac{\rho_1}{\delta}$.

Theorem 4.3. *Suppose that assumptions* $(\widehat{H}_1)_0-(\widehat{H}_3)_0$ *hold and furthermore* $N \le \min\{M_1, M_3\}$. *Then the boundary value problem* (4.1) − (4.3) *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that* $x_1(r_3) > M_3r_3$, $x_1(r_2) < \frac{\rho}{\delta}$, $x_2(r_2) > \frac{\rho}{\delta}$ *and* $x_2(r_1) < M_1r_1$.

Also, the following corollary corresponds to Corollary 3.3.

Corollary 4.4. *Suppose that assumptions* $(H_1)_0-(H_3)_0$ *(resp.* $(\widehat{H}_1)_0-(\widehat{H}_3)_0$) *hold and furthermore* $N \le M$ *(resp.* $N \le \min\{M_1, M_3\}$). *Then the boundary value problem* (4.1) − (4.3) *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that*

$$\frac{\rho_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\rho_1}{\delta}$$

(resp. $M_3 < \|x_1\| < \frac{\rho}{\delta} < \|x_2\| < M_1$).

5. Applications

1. Consider the boundary value problem

$$x''(t) + \left(x(t-\frac{1}{2}) - \frac{4}{5}\right)^5 + 1 = 0, \quad t \in I := [0,1] \tag{5.1}$$

$$x_0(t) = \phi(t) := t^2, \quad t \in J := [-\frac{1}{2}, 0] \tag{5.2}$$

and

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{5.3}$$

where $g(t) = \frac{1}{2}t$, $t \in I$.

Obviously, $f(t,y) := (y - \frac{4}{5})^5 + 1$ and ϕ is nonnegative on $\mathbb{R}^+ \times C^+(J)$,$\phi$ is nonnegative on $\mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{1}{2} > 0$. Set $r_1 = \frac{2}{5}$, $r_2 = \frac{3}{5}$ and $r_3 = \frac{4}{5}$. Define $L(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequality

$$L(M) < \frac{5}{6}M$$

holds for $M = \frac{7}{5}$, assumption (H_1) is satisfied.

Additionally, set $\delta = \frac{1}{4}$, $\tau(t) = \frac{1}{2}$, $t \in I$, $v(t) = 1$, $t \in I$ and $w(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$. Then, $X = [\frac{3}{4}, 1]$ and the inequalities in assumption (H_3) take the forms

$$w(\rho_1) > \frac{64}{3}\rho_1 \text{ and } w(\rho_3) > \frac{32}{3}\rho_3,$$

which are satisfied for $\rho_1 = 6$ and $\rho_3 = \frac{1}{20}$. Finally, it is obvious that $\|\phi\|_J \leq \frac{7}{5}$, so we can apply Theorem 3.1 to get that the boundary value problem (5.1) – (5.3) has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2, such that

$$x_1(\frac{4}{5}) > \frac{1}{5},\ x_1(\frac{3}{5}) < \frac{21}{25},\ x_2(\frac{3}{5}) > \frac{21}{25} \text{ and } x_2(\frac{2}{5}) < 24.$$

2. Once again, consider the boundary value problem (5.1) – (5.3). As mentioned in Application 1, $f(t,y) := (y - \frac{4}{5})^5 + 1$ is nonnegative on $\mathbb{R}^+ \times C^+(J)$, ϕ is nonnegative on $\mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{1}{2} > 0$.

Having in mind Corollary 3.3, we set $r_1 = r_2 = r_3 = 1$. Define $L(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequality

$$L(M) < M$$

holds for $M = \frac{11}{10}$, assumption (H_1) is satisfied.

Additionally, set $\delta = \frac{1}{4}$, $\tau(t) = \frac{1}{2}$, $t \in I$, $v(t) = 1$, $t \in I$ and $w(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$. Then, $X = [\frac{3}{4}, 1]$ and the inequalities in assumption (H_3) take the forms

$$w(\rho_1) > \frac{64}{7}\rho_1 \text{ and } w(\rho_3) > \frac{64}{7}\rho_3,$$

which are satisfied for $\rho_1 = \frac{27}{10}$ and $\rho_3 = \frac{1}{12}$. Finally, it is obvious that $\|\phi\|_J \leq \frac{11}{10}$, so we can apply Corollary 3.3 to get that the boundary value problem (5.1) – (5.3) has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2, such that

$$\frac{1}{3} < \|x_1\| < \frac{11}{10} < \|x_2\| < \frac{54}{5}.$$

3. Consider the boundary value problem

$$x''(t) + 8\arctan(10x(t) - 14) + 12 = 0,\ \ t \in I := [0,1] \tag{5.4}$$

$$x(0) = 0, \tag{5.5}$$

and

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{5.6}$$

where $g(t) = \frac{1}{4}t$, $t \in I$.

Obviously, $f(t,y) := 8\arctan(10y - 14) + 12$ is positive on $\mathbb{R}^+ \times \mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{3}{4} > 0$. Set $r_1 = \frac{1}{4}$, $r_2 = \frac{1}{2}$ and $r_3 = \frac{3}{4}$. Define $L(t) = 8\arctan(10t - 14) + 12$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequalities

$$L(M_1) < \frac{24}{25}M_1 \text{ and } L(M_3) < \frac{24}{19}M_3$$

hold for $M_1 = 30$ and $M_3 = \frac{1}{100}$, assumption $(\widehat{H}_1)_0$ is satisfied.

Additionally, set $\delta = \frac{1}{2}$, $v(t) = 1$, $t \in I$ and $w(t) = 8\arctan(10t - 14) + 12$, $t \in \mathbb{R}^+$. Then, $Z = [\frac{1}{2}, 1]$ and assumption $(\widehat{H}_3)_0$ takes the form

$$w(\rho) > \frac{96}{15}\rho,$$

which is satisfied for $\rho = \frac{7}{5}$. Finally, it is obvious that $N = 0 \leq \frac{1}{100} = \min\{M_1, M_3\}$, so we can apply Theorem 4.3 to get that the boundary value problem (5.4) – (5.6) has at least two concave and nondecreasing and positive on I solutions x_1, x_2, such that

$$x_1(\frac{3}{4}) > \frac{3}{400}, \ x_1(\frac{1}{2}) < \frac{14}{5}, \ x_2(\frac{1}{2}) > \frac{14}{5} \text{ and } x_2(\frac{1}{4}) < \frac{15}{2}.$$

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